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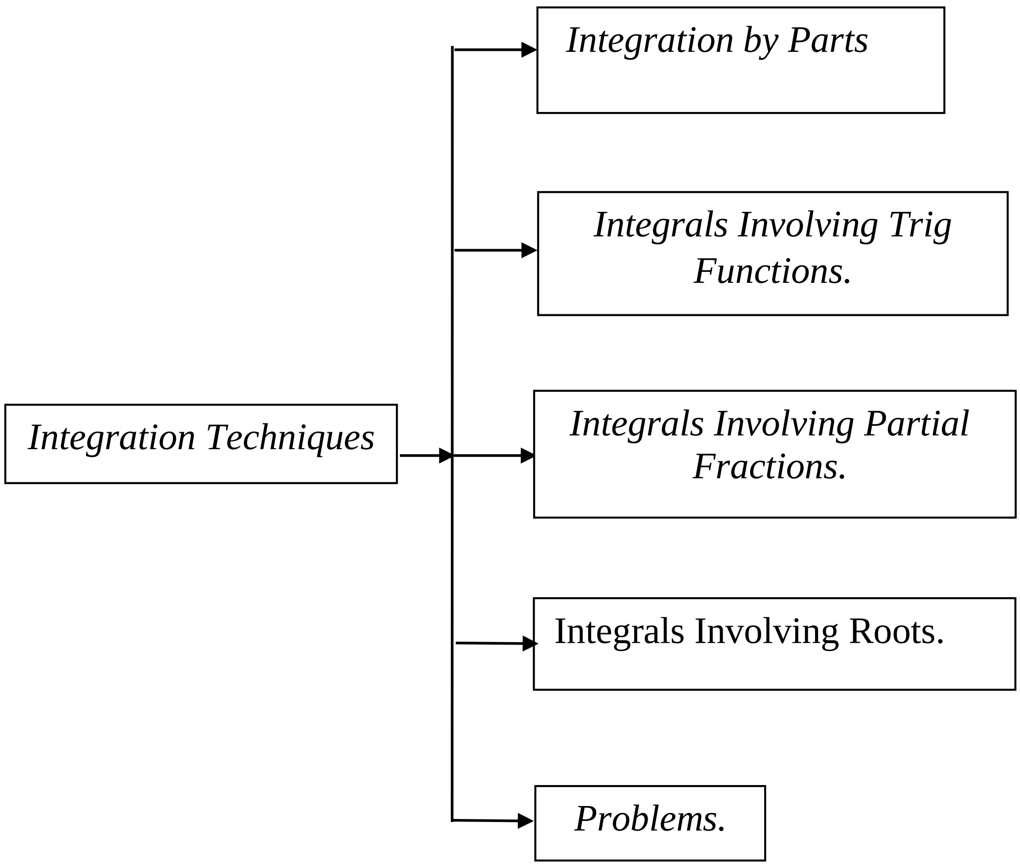
Lecture – One

Integration Techniques



Integration Techniques:-

*1- **Integration by Parts***

So let's derive the integration by parts formula. We'll start with the product rule.

$$(f\,g)' = f'\,g + f\,g'$$

Now, integrate both sides of this.

$$\int (f\,g)'\,dx = \int f'\,g + f\,g'\,dx$$

The left side is easy enough to integrate and we'll split up the right side of the integral.

$$fg = \int f'\,g\,dx + \int f\,g'\,dx$$

Finally, rewrite the formula as follows and we arrive at the integration by parts formula.

$$\int f\,g'\,dx = fg - \int f'\,g\,dx$$

This is not the easiest formula to use however. So, let's do a couple of substitutions.

$$u = f(x) \qquad\qquad v = g(x)$$

$$du = f'(x)\,dx \qquad\qquad dv = g'(x)\,dx$$

Using these substitutions gives us the formula that most people think of as the integration by parts formula.

$$\int u\,dv = uv - \int v\,du$$

To use this formula we will need to identify u and dv, compute du and v and then use the formula. Note as well that computing v is very easy. All we need to do is integrate dv.

$$v = \int dv$$

Example 1 Evaluate the following integral

$$\int x\mathbf{e}^{6x}\,dx$$

Solution

$$u = x \qquad\qquad dv = \mathbf{e}^{6x}\,dx$$

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$$du = dx \qquad\qquad v = \int \mathbf{e}^{6x}\,dx = \frac{1}{6}\mathbf{e}^{6x}$$

Next, let's take a look at integration by parts for definite integrals. The integration by parts formula for definite integrals is,

Integration by Parts, Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example 2 Evaluate the following integral.

$$\int_{-1}^{2} x\mathbf{e}^{6x}\,dx$$

Solution
This is the same integral that we looked at in the first example so we'll use the same u and dv to get,

$$\begin{aligned}\int_{-1}^{2} x\mathbf{e}^{6x}\,dx &= \left.\frac{x}{6}\mathbf{e}^{6x}\right|_{-1}^{2} - \frac{1}{6}\int_{-1}^{2}\mathbf{e}^{6x}\,dx \\ &= \left.\frac{x}{6}\mathbf{e}^{6x}\right|_{-1}^{2} - \left.\frac{1}{36}\mathbf{e}^{6x}\right|_{-1}^{2} \\ &= \frac{11}{36}\mathbf{e}^{12} + \frac{7}{36}\mathbf{e}^{-6}\end{aligned}$$

Example 3 Evaluate the following integral.

$$\int (3t+5)\cos\left(\frac{t}{4}\right)dt$$

Solution
Instead of splitting the integral up let's instead use the following choices for u and dv.

$$u = 3t+5 \qquad dv = \cos\left(\frac{t}{4}\right)dt$$

$$du = 3\,dt \qquad v = 4\sin\left(\frac{t}{4}\right)$$

The integral is then,

$$\begin{aligned}\int (3t+5)\cos\left(\frac{t}{4}\right)dt &= 4(3t+5)\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right)dt \\ &= 4(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + c\end{aligned}$$

Example 4 Evaluate the following integral.

$$\int w^2 \sin(10w)\,dw$$

Solution
For this example we'll use the following choices for u and dv.

$$u = w^2 \qquad dv = \sin(10w)\,dw$$

$$du = 2w\,dw \qquad v = -\frac{1}{10}\cos(10w)$$

The integral is then,

$$\int w^2 \sin(10w)\,dw = -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\int w\cos(10w)\,dw$$

In this example, unlike the previous examples, the new integral will also require integration by parts. For this second integral we will use the following choices.

$$u = w \qquad dv = \cos(10w)\,dw$$

$$du = dw \qquad v = \frac{1}{10}\sin(10w)$$

So, the integral becomes,

$$\begin{aligned}\int w^2 \sin(10w)\,dw &= -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) - \frac{1}{10}\int \sin(10w)\,dw\right)\\ &= -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) + \frac{1}{100}\cos(10w)\right) + c\\ &= -\frac{w^2}{10}\cos(10w) + \frac{w}{50}\sin(10w) + \frac{1}{500}\cos(10w) + c\end{aligned}$$

Example 5 Evaluate the following integral

$$\int x\sqrt{x+1}\,dx$$

(a) Using Integration by Parts. [Solution]
(b) Using a standard Calculus I substitution. [Solution]

Solution

(a) Evaluate using Integration by Parts.

In this case we'll use the following choices for u and dv.

$$u = x \qquad dv = \sqrt{x+1}\,dx$$

$$du = dx \qquad v = \frac{2}{3}(x+1)^{\frac{3}{2}}$$

The integral is then,

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}\,dx\\ &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + c\end{aligned}$$

(b) Evaluate Using a standard Calculus I substitution.

Now let's do the integral with a substitution. We can use the following substitution.

$$u = x + 1 \qquad\qquad x = u - 1 \qquad\qquad du = dx$$

Notice that we'll actually use the substitution twice, once for the quantity under the square root and once for the x in front of the square root. The integral is then,

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int (u-1)\sqrt{u}\,du \\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c\end{aligned}$$

Example 6 Evaluate the following integral.

$$\int \mathbf{e}^{\theta}\cos\theta\,d\theta$$

Solution

$$u = \cos\theta \qquad\qquad dv = \mathbf{e}^{\theta}\,d\theta$$

$$du = -\sin\theta\,d\theta \qquad\qquad v = \mathbf{e}^{\theta}$$

The integral is then,

$$\int \mathbf{e}^{\theta}\cos\theta\,d\theta = \mathbf{e}^{\theta}\cos\theta + \int \mathbf{e}^{\theta}\sin\theta\,d\theta$$

So, it looks like we'll do integration by parts again. Here are our choices this time.

$$u = \sin\theta \qquad\qquad dv = \mathbf{e}^{\theta}\,d\theta$$

$$du = \cos\theta\,d\theta \qquad\qquad v = \mathbf{e}^{\theta}$$

The integral is now,

$$\int \mathbf{e}^{\theta}\cos\theta\,d\theta = \mathbf{e}^{\theta}\cos\theta + \mathbf{e}^{\theta}\sin\theta - \int \mathbf{e}^{\theta}\cos\theta\,d\theta$$

$$2\int \mathbf{e}^{\theta}\cos\theta\,d\theta = \mathbf{e}^{\theta}\cos\theta + \mathbf{e}^{\theta}\sin\theta$$

All we need to do now is divide by 2 and we're done. The integral is,

$$\int \mathbf{e}^{\theta}\cos\theta\,d\theta = \frac{1}{2}\left(\mathbf{e}^{\theta}\cos\theta + \mathbf{e}^{\theta}\sin\theta\right) + c$$

2- ***Integrals Involving Trig Functions.***

Let's start off with an integral that we should already be able to do.

$$\int \cos x \sin^5 x \, dx = \int u^5 \, du \qquad \text{using the substitution } u = \sin x$$
$$= \frac{1}{6}\sin^6 x + c$$

Example 1 Evaluate the following integral.

$$\int \sin^5 x \, dx$$

Solution

This integral no longer has the cosine in it that would allow us to use the substitution that we used above. Therefore, that substitution won't work and we are going to have to find another way of doing this integral.

Let's first notice that we could write the integral as follows,

$$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx = \int \left(\sin^2 x\right)^2 \sin x \, dx$$

Now recall the trig identity,

$$\cos^2 x + \sin^2 x = 1 \qquad \Rightarrow \qquad \sin^2 x = 1 - \cos^2 x$$

With this identity the integral can be written as,

$$\int \sin^5 x \, dx = \int \left(1 - \cos^2 x\right)^2 \sin x \, dx$$

and we can now use the substitution $u = \cos x$. Doing this gives us,

$$\begin{aligned} \int \sin^5 x \, dx &= -\int \left(1 - u^2\right)^2 du \\ &= -\int 1 - 2u^2 + u^4 \, du \\ &= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + c \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c \end{aligned}$$

The exponent on the remaining sines will then be even and we can easily convert the remaining sines to cosines using the identity,

$$\cos^2 x + \sin^2 x = 1 \qquad (1)$$

Example 2 Evaluate the following integral.

$$\int \sin^6 x \cos^3 x \, dx$$

Solution

So, in this case we've got both sines and cosines in the problem and in this case the exponent on the sine is even while the exponent on the cosine is odd. So, we can use a similar technique in this integral. This time we'll strip out a cosine and convert the rest to sines.

$$\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x \cos^2 x \cos x \, dx$$

$$= \int \sin^6 x \left(1 - \sin^2 x\right) \cos x \, dx \qquad u = \sin x$$

$$= \int u^6 \left(1 - u^2\right) du$$

$$= \int u^6 - u^8 \, du$$

$$= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + c$$

At this point let's pause for a second to summarize what we've learned so far about integrating powers of sine and cosine.

$$\int \sin^n x \cos^m x \, dx \tag{2}$$

The integrals involving products of sines and cosines in which both exponents are even can be done using one or more of the following formulas to rewrite the integrand.

$$\cos^2 x = \frac{1}{2}\left(1 + \cos(2x)\right)$$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos(2x)\right)$$

$$\sin x \cos x = \frac{1}{2} \sin(2x)$$

Example 3 Evaluate the following integral.

$$\int \sin^2 x \cos^2 x \, dx$$

Solution

Solution 1

In this solution we will use the two half angle formulas above and just substitute them into the integral.

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}\left(1 - \cos(2x)\right)\left(\frac{1}{2}\right)\left(1 + \cos(2x)\right) dx$$

$$= \frac{1}{4} \int 1 - \cos^2(2x) \, dx$$

In fact to eliminate the remaining problem term all that we need to do is reuse the first half angle formula given above.

$$\begin{aligned}\int \sin^2 x \cos^2 x\,dx &= \frac{1}{4}\int 1-\frac{1}{2}\left(1+\cos(4x)\right)dx \\ &= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos(4x)\,dx \\ &= \frac{1}{4}\left(\frac{1}{2}x-\frac{1}{8}\sin(4x)\right)+c \\ &= \frac{1}{8}x-\frac{1}{32}\sin(4x)+c\end{aligned}$$

Solution 2

In this solution we will use the half angle formula to help simplify the integral as follows.

$$\begin{aligned}\int \sin^2 x \cos^2 x\,dx &= \int (\sin x \cos x)^2\,dx \\ &= \int \left(\frac{1}{2}\sin(2x)\right)^2 dx \\ &= \frac{1}{4}\int \sin^2(2x)\,dx\end{aligned}$$

Now, we use the double angle formula for sine to reduce to an integral that we can do.

$$\begin{aligned}\int \sin^2 x \cos^2 x\,dx &= \frac{1}{8}\int 1-\cos(4x)\,dx \\ &= \frac{1}{8}x-\frac{1}{32}\sin(4x)+c\end{aligned}$$

Sometimes in the process of reducing integrals in which both exponents are even we will run across products of sine and cosine in which the arguments are different. These will require one of the following formulas to reduce the products to integrals that we can do.

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$

Example 4 Evaluate the following integral.

$$\int \cos(15x)\cos(4x)\,dx$$

Solution

This integral requires the last formula listed above.

$$\begin{aligned}\int \cos(15x)\cos(4x)\,dx &= \frac{1}{2}\int \cos(11x)+\cos(19x)\,dx \\ &= \frac{1}{2}\left(\frac{1}{11}\sin(11x)+\frac{1}{19}\sin(19x)\right)+c\end{aligned}$$

It’s now time to look at integrals that involve products of secants and tangents. This time, let’s do a little analysis of the possibilities before we just jump into examples. The general integral will be,

$$\int \sec^n x \tan^m x \, dx \tag{3}$$

The first thing to notice is that we can easily convert even powers of secants to tangents and even powers of tangents to secants by using a formula similar to (1). In fact, the formula can be derived from (1) so let’s do that.

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x \tag{4}$$

Now, we’re going to want to deal with (3) similarly to how we dealt with (2). We’ll want to eventually use one of the following substitutions.

$$u = \tan x \qquad du = \sec^2 x \, dx$$

$$u = \sec x \qquad du = \sec x \tan x \, dx$$

Example 5 Evaluate the following integral.

$$\int \sec^9 x \tan^5 x \, dx$$

Solution

First note that since the exponent on the secant isn’t even we can’t use the substitution $u = \tan x$. However, the exponent on the tangent is odd and we’ve got a secant in the integral and so we will be able to use the substitution $u = \sec x$. This means stripping out a single tangent (along with a secant) and converting the remaining tangents to secants using (4).

Here’s the work for this integral.

$$\begin{aligned}
\int \sec^9 x \tan^5 x \, dx &= \int \sec^8 x \tan^4 x \tan x \sec x \, dx \\
&= \int \sec^8 x \left(\sec^2 x - 1\right)^2 \tan x \sec x \, dx \qquad\qquad u = \sec x \\
&= \int u^8 \left(u^2 - 1\right)^2 du \\
&= \int u^{12} - 2u^{10} + u^8 \, du \\
&= \frac{1}{13}\sec^{13} x - \frac{2}{11}\sec^{11} x + \frac{1}{9}\sec^9 x + c
\end{aligned}$$

Example 6 Evaluate the following integral.

$$\int \sec^4 x \tan^6 x \, dx$$

Solution

So, in this example the exponent on the tangent is even so the substitution $u = \sec x$ won't work. The exponent on the secant is even and so we can use the substitution $u = \tan x$ for this integral. That means that we need to strip out two secants and convert the rest to tangents. Here is the work for this integral.

$$\begin{aligned}
\int \sec^4 x \tan^6 x \, dx &= \int \sec^2 x \tan^6 x \sec^2 x \, dx \\
&= \int \left(\tan^2 x + 1\right) \tan^6 x \sec^2 x \, dx \qquad u = \tan x \\
&= \int \left(u^2 + 1\right) u^6 \, du \\
&= \int u^8 + u^6 \, du \\
&= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + c
\end{aligned}$$

Example 7 Evaluate the following integral.

$$\int \tan x \, dx$$

Solution

To do this integral all we need to do is recall the definition of tangent in terms of sine and cosine and then this integral is nothing more than a Calculus I substitution.

$$\begin{aligned}
\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \qquad u = \cos x \\
&= -\int \frac{1}{u} \, du \\
&= -\ln\left|\cos x\right| + c \qquad r \ln x = \ln x^r \\
&= \ln\left|\cos x\right|^{-1} + c \\
&\ln\left|\sec x\right| + c
\end{aligned}$$

Example 8 Evaluate the following integral.

$$\int \tan^3 x \, dx$$

Solution

The trick to this one is do the following manipulation of the integrand.

$$\begin{aligned}
\int \tan^3 x \, dx &= \int \tan x \tan^2 x \, dx \\
&= \int \tan x \left(\sec^2 x - 1\right) dx \\
&= \int \tan x \sec^2 x \, dx - \int \tan x \, dx
\end{aligned}$$

We can now use the substitution $u = \tan x$ on the first integral and the results from the previous example on the second integral.

The integral is then,

$$\int \tan^3 x \, dx = \frac{1}{2}\tan^2 x - \ln\left|\sec x\right| + c$$

Note that all odd powers of tangent (with the exception of the first power) can be integrated using the same method we used in the previous example. For instance,

$$\int \tan^5 x \, dx = \int \tan^3 x\left(\sec^2 x - 1\right) dx = \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$$

Example 9 Evaluate the following integral.

$$\int \sec x \, dx$$

Solution

This one isn't too bad once you see what you've got to do. By itself the integral can't be done. However, if we manipulate the integrand as follows we can do it.

$$\begin{aligned} \int \sec x \, dx &= \int \frac{\sec x \left(\sec x + \tan x \right)}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x} dx \end{aligned}$$

In this form we can do the integral using the substitution $u = \sec x + \tan x$. Doing this gives,

$$\int \sec x \, dx = \ln \left| \sec x + \tan x \right| + c$$

Example 10 Evaluate the following integral.

$$\int \sec^3 x \, dx$$

Solution

This one is different from any of the other integrals that we've done in this section. The first step to doing this integral is to perform integration by parts using the following choices for u and dv.

$$\begin{aligned} u &= \sec x & dv &= \sec^2 x \, dx \\ du &= \sec x \tan x \, dx & v &= \tan x \end{aligned}$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

To do this integral we'll first write the tangents in the integral in terms of secants. Again, this is not necessarily an obvious choice but it's what we need to do in this case.

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \sec x \left(\sec^2 x - 1 \right) dx \\ &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \end{aligned}$$

the first integral is exactly the integral we're being asked to evaluate with a minus sign in front. So, add it to both sides to get,

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln \left| \sec x + \tan x \right|$$

Finally divide by two and we're done.

$$\int \sec^3 x \, dx = \frac{1}{2} \left(\sec x \tan x + \ln \left| \sec x + \tan x \right| \right) + c$$

3- ***Integrals Involving Partial Fractions.***

let's start this section out with an integral that we can already do so we can contrast it with the integrals that we'll be doing in this section.

$$\int \frac{2x-1}{x^2-x-6}\,dx = \int \frac{1}{u}\,du \qquad \text{using} \quad u = x^2-x-6 \quad \text{and} \quad du = (2x-1)\,dx$$

$$= \ln\left|x^2-x-6\right| + c$$

So, if the numerator is the derivative of the denominator (or a constant multiple of the derivative of the denominator) doing this kind of integral is fairly simple. However, often the numerator isn't the derivative of the denominator (or a constant multiple). For example, consider the following integral.

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

This process of taking a rational expression and decomposing it into simpler rational expressions that we can add or subtract to get the original rational expression is called **partial fraction decomposition**. Many integrals involving rational expressions can be done if we first do partial fractions on the integrand.

So, let's do a quick review of partial fractions. We'll start with a rational expression in the form,

$$f(x) = \frac{P(x)}{Q(x)}$$

where both $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is smaller than the degree of $Q(x)$. Recall that the degree of a polynomial is the largest exponent in the polynomial. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

So, once we've determined that partial fractions can be done we factor the denominator as completely as possible. Then for each factor in the denominator we can use the following table to determine the term(s) we pick up in the partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}, \quad k = 1,2,3,\ldots$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}, \quad k = 1,2,3,\ldots$

There are several methods for determining the coefficients for each term and we will go over each of those in the following examples.
Let's start the examples by doing the integral above.

Example 1 Evaluate the following integral.

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

Solution

The first step is to factor the denominator as much as possible and get the form of the partial fraction decomposition. Doing this gives,

$$\frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

The next step is to actually add the right side back up.

$$\frac{3x+11}{(x-3)(x+2)} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

Now, we need to choose A and B so that the numerators of these two are equal for every x. To do this we'll need to set the numerators equal.

$$3x+11 = A(x+2)+B(x-3)$$

What we're going to do here is to notice that the numerators must be equal for *any* x that we would choose to use. In particular the numerators must be equal for $x=-2$ and $x=3$. So, let's plug these in and see what we get.

$$\begin{array}{llll} x=-2 & 5 = A(0)+B(-5) & \Rightarrow & B=-1 \\ x=3 & 20 = A(5)+B(0) & \Rightarrow & A=4 \end{array}$$

At this point there really isn't a whole lot to do other than the integral.

$$\begin{aligned} \int \frac{3x+11}{x^2-x-6}\,dx &= \int \frac{4}{x-3} - \frac{1}{x+2}\,dx \\ &= \int \frac{4}{x-3}\,dx - \int \frac{1}{x+2}\,dx \\ &= 4\ln|x-3| - \ln|x+2| + c \end{aligned}$$

There is also another integral that often shows up in these kinds of problems so we may as well give the formula for it here since we are already on the subject.

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Example 2 Evaluate the following integral.

$$\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx$$

Solution

We won't be putting as much detail into this solution as we did in the previous example. The first thing is to factor the denominator and get the form of the partial fraction decomposition.

$$\frac{x^2+4}{x(x+2)(3x-2)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{3x-2}$$

The next step is to set numerators equal. If you need to actually add the right side together to get

$$x^2+4 = A(x+2)(3x-2) + Bx(3x-2) + Cx(x+2)$$

As with the previous example it looks like we can just pick a few values of x and find the constants so let's do that.

$$x=0 \qquad 4 = A(2)(-2) \qquad \Rightarrow \qquad A=-1$$

$$x=-2 \qquad 8 = B(-2)(-8) \qquad \Rightarrow \qquad B=\frac{1}{2}$$

$$x=\frac{2}{3} \qquad \frac{40}{9} = C\left(\frac{2}{3}\right)\left(\frac{8}{3}\right) \qquad \Rightarrow \qquad C=\frac{40}{16}=\frac{5}{2}$$

Now, let's do the integral.

$$\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx = \int -\frac{1}{x} + \frac{\frac{1}{2}}{x+2} + \frac{\frac{5}{2}}{3x-2}\,dx$$

$$= -\ln|x| + \frac{1}{2}\ln|x+2| + \frac{5}{6}\ln|3x-2| + c$$

Example 3 Evaluate the following integral.

$$\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}\,dx$$

Solution

This time the denominator is already factored so let's just jump right to the partial fraction decomposition.

$$\frac{x^2-29x+5}{(x-4)^2(x^2+3)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+3}$$

Setting numerators gives,

$$x^2-29x+5 = A(x-4)(x^2+3) + B(x^2+3) + (Cx+D)(x-4)^2$$

In this case we aren't going to be able to just pick values of x that will give us all the constants. Therefore, we will need to work this the second (and often longer) way. The first step is to multiply out the right side and collect all the like terms together. Doing this gives,

$$x^2-29x+5 = (A+C)x^3 + (-4A+B-8C+D)x^2 + (3A+16C-8D)x - 12A + 3B + 16D$$

In other words we will need to set the coefficients of like powers of x equal. This will give a system of equations that can be solved.

$$\left.\begin{aligned}
x^3 &: & A+C&=0\\
x^2 &: & -4A+B-8C+D&=1\\
x^1 &: & 3A+16C-8D&=-29\\
x^0 &: & -12A+3B+16D&=5
\end{aligned}\right\}\quad\Rightarrow\quad A=1, B=-5, C=-1, D=2$$

Now, let's take a look at the integral.

$$\begin{aligned}
\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}dx &= \int \frac{1}{x-4}-\frac{5}{(x-4)^2}+\frac{-x+2}{x^2+3}dx\\
&= \int \frac{1}{x-4}-\frac{5}{(x-4)^2}-\frac{x}{x^2+3}+\frac{2}{x^2+3}dx\\
&= \ln|x-4|+\frac{5}{x-4}-\frac{1}{2}\ln|x^2+3|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c
\end{aligned}$$

Example 4 Evaluate the following integral.

$$\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}dx$$

Solution

Let's first get the general form of the partial fraction decomposition.

$$\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}$$

Now, set numerators equal, expand the right side and collect like terms.

$$\begin{aligned}
x^3+10x^2+3x+36 &= A(x^2+4)^2+(Bx+C)(x-1)(x^2+4)+(Dx+E)(x-1)\\
&= (A+B)x^4+(C-B)x^3+(8A+4B-C+D)x^2+\\
&\qquad (-4B+4C-D+E)x+16A-4C-E
\end{aligned}$$

Setting coefficient equal gives the following system.

$$\left.\begin{aligned}
x^4 &: & A+B&=0\\
x^3 &: & C-B&=1\\
x^2 &: & 8A+4B-C+D&=10\\
x^1 &: & -4B+4C-D+E&=3\\
x^0 &: & 16A-4C-E&=36
\end{aligned}\right\}\quad\Rightarrow\quad A=2, B=-2, C=-1, D=1, E=0$$

Here's the integral.

$$\begin{aligned}\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,dx &= \int \frac{2}{x-1}+\frac{-2x-1}{x^2+4}+\frac{x}{(x^2+4)^2}\,dx \\ &= \int \frac{2}{x-1}-\frac{2x}{x^2+4}-\frac{1}{x^2+4}+\frac{x}{(x^2+4)^2}\,dx \\ &= 2\ln|x-1|-\ln|x^2+4|-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{2}\frac{1}{x^2+4}+c\end{aligned}$$

To this point we've only looked at rational expressions where the degree of the numerator was strictly less that the degree of the denominator. Of course not all rational expressions will fit into this form and so we need to take a look at a couple of examples where this isn't the case.

If a rational function $\frac{R(x)}{Q(x)}$ is such that the degree of $R(x)$ is **greater than** the degree of $Q(x)$, then one must use long division and write the rational function in the form

$$\frac{R(x)}{Q(x)} = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n + \frac{P(x)}{Q(x)}$$

where now $P(x)$ is a remainder term with the degree of $P(x)$ **less than** the degree of $Q(x)$ and our object is to integrate each term of the above representation.

Example 5 Evaluate the following integral.

$$\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,dx$$

Solution

So, in this case the degree of the numerator is 4 and the degree of the denominator is 3. Therefore, partial fractions can't be done on this rational expression.

To fix this up we'll need to do long division on this to get it into a form that we can deal with. Here is the work for that.

$$\begin{array}{r} x-2 \\ x^3-3x^2\overline{\big)\,x^4-5x^3+6x^2-18} \\ \underline{-\left(x^4-3x^3\right)\qquad\qquad} \\ -2x^3+6x^2-18 \\ \underline{-\left(-2x^3+6x^2\right)\quad} \\ -18 \end{array}$$

$$\frac{x^4-5x^3+6x^2-18}{x^3-3x^2}=x-2-\frac{18}{x^3-3x^2}$$

and the integral becomes,

$$\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,dx=\int x-2-\frac{18}{x^3-3x^2}\,dx$$
$$=\int x-2\,dx-\int \frac{18}{x^3-3x^2}\,dx$$

The first integral we can do easily enough and the second integral is now in a form that allows us to do partial fractions. So, let's get the general form of the partial fractions for the second integrand.

$$\frac{18}{x^2\left(x-3\right)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-3}$$

Setting numerators equal gives us,

$$18=Ax\left(x-3\right)+B\left(x-3\right)+Cx^2$$

$x=0$	$18=B\left(-3\right)$	$\Rightarrow$	$B=-6$
$x=3$	$18=C\left(9\right)$	$\Rightarrow$	$C=2$
$x=1$	$18=A\left(-2\right)+B\left(-2\right)+C=-2A+14$	$\Rightarrow$	$A=-2$

The integral is then,

$$\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,dx=\int x-2\,dx-\int -\frac{2}{x}-\frac{6}{x^2}+\frac{2}{x-3}\,dx$$
$$=\frac{1}{2}x^2-2x+2\ln\left|x\right|-\frac{6}{x}-2\ln\left|x-3\right|+c$$

4- ***Integrals Involving Roots.***

Example 1 Evaluate the following integral.

$$\int \frac{x+2}{\sqrt[3]{x-3}}\,dx$$

Solution

Sometimes when faced with an integral that contains a root we can use the following substitution to simplify the integral into a form that can be easily worked with.

$$u = \sqrt[3]{x-3}$$

So, instead of letting u be the stuff under the radical as we often did in Calculus I we let u be the whole radical. Now, there will be a little more work here since we will also need to know what x is so we can substitute in for that in the numerator and so we can compute the differential, dx. This is easy enough to get however. Just solve the substitution for x as follows,

$$x = u^3 + 3 \qquad\qquad dx = 3u^2\,du$$

Using this substitution the integral is now,

$$\begin{aligned}\int \frac{\left(u^3+3\right)+2}{u}3u^2du &= \int 3u^4 + 15u\,du \\ &= \frac{3}{5}u^5 + \frac{15}{2}u^2 + c \\ &= \frac{3}{5}(x-3)^{\frac{5}{3}} + \frac{15}{2}(x-3)^{\frac{2}{3}} + c\end{aligned}$$

So, sometimes, when an integral contains the root $\sqrt[n]{g(x)}$ the substitution,

$$u = \sqrt[n]{g(x)}$$

can be used to simplify the integral into a form that we can deal with.

Example 2 Evaluate the following integral.

$$\int \frac{2}{x - 3\sqrt{x+10}}\,dx$$

Solution

We'll do the same thing we did in the previous example. Here's the substitution and the extra work we'll need to do to get x in terms of u.

$$u = \sqrt{x+10} \qquad\qquad x = u^2 - 10 \qquad\qquad dx = 2u\,du$$

With this substitution the integral is,

$$\int \frac{2}{x - 3\sqrt{x+10}}\,dx = \int \frac{2}{u^2 - 10 - 3u}(2u)\,du = \int \frac{4u}{u^2 - 3u - 10}\,du$$

This integral can now be done with partial fractions.

$$\frac{4u}{(u-5)(u+2)} = \frac{A}{u-5} + \frac{B}{u+2}$$

Setting numerators equal gives,

$$4u = A(u+2) + B(u-5)$$

Picking value of u gives the coefficients.

$$u = -2 \qquad -8 = B(-7) \qquad B = \frac{8}{7}$$

$$u = 5 \qquad 20 = A(7) \qquad A = \frac{20}{7}$$

The integral is then,

$$\begin{aligned}\int \frac{2}{x - 3\sqrt{x+10}}\,dx &= \int \frac{\frac{20}{7}}{u-5} + \frac{\frac{8}{7}}{u+2}\,du \\ &= \frac{20}{7}\ln|u-5| + \frac{8}{7}\ln|u+2| + c \\ &= \frac{20}{7}\ln\left|\sqrt{x+10} - 5\right| + \frac{8}{7}\ln\left|\sqrt{x+10} + 2\right| + c\end{aligned}$$

5- Problems.

A- *Integration by Parts*

Evaluate each of the following integrals.

1. $\int 4x\cos(2-3x)\,dx$

2. $\int_{6}^{0}(2+5x)\mathbf{e}^{\frac{1}{3}x}\,dx$

3. $\int(3t+t^{2})\sin(2t)\,dt$

4. $\int 6\tan^{-1}\left(\frac{8}{w}\right)dw$

5. $\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)dz$

6. $\int_{0}^{\pi}x^{2}\cos(4x)\,dx$

7. $\int t^{7}\sin(2t^{4})\,dt$

8. $\int y^{6}\cos(3y)\,dy$

9. $\int(4x^{3}-9x^{2}+7x+3)\mathbf{e}^{-x}\,dx$

B-

Integrals Involving Trig Functions

Evaluate each of the following integrals.

1. $\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)dx$

2. $\int \sin^8(3z)\cos^5(3z)\,dz$

3. $\int \cos^4(2t)\,dt$

4. $\int_{\pi}^{2\pi} \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)dw$

5. $\int \sec^6(3y)\tan^2(3y)\,dy$

6. $\int \tan^3(6x)\sec^{10}(6x)\,dx$

7. $\int_0^{\frac{\pi}{4}} \tan^7(z)\sec^3(z)\,dz$

8. $\int \cos(3t)\sin(8t)\,dt$

C-

Partial Fractions

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{4}{x^2 + 5x - 14}\,dx$

2. $\displaystyle\int \frac{8 - 3t}{10t^2 + 13t - 3}\,dt$

3. $\displaystyle\int_{-1}^{0} \frac{w^2 + 7w}{(w+2)(w-1)(w-4)}\,dw$

4. $\displaystyle\int \frac{8}{3x^3 + 7x^2 + 4x}\,dx$

5. $\displaystyle\int_{2}^{4} \frac{3z^2 + 1}{(z+1)(z-5)^2}\,dz$

6. $\displaystyle\int \frac{4x - 11}{x^3 - 9x^2}\,dx$

7. $\displaystyle\int \frac{z^2 + 2z + 3}{(z-6)(z^2+4)}\,dz$

8. $\displaystyle\int \frac{8 + t + 6t^2 - 12t^3}{(3t^2+4)(t^2+7)}\,dt$

D-

Integrals Involving Roots

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{7}{2+\sqrt{x-4}}\,dx$

2. $\displaystyle\int \frac{1}{w+2\sqrt{1-w}+2}\,dw$

3. $\displaystyle\int \frac{t-2}{t-3\sqrt{2t-4}+2}\,dt$

First year/ 2nd Semester - 2018-2019- Chemical and Petroleum Engineering Department

By

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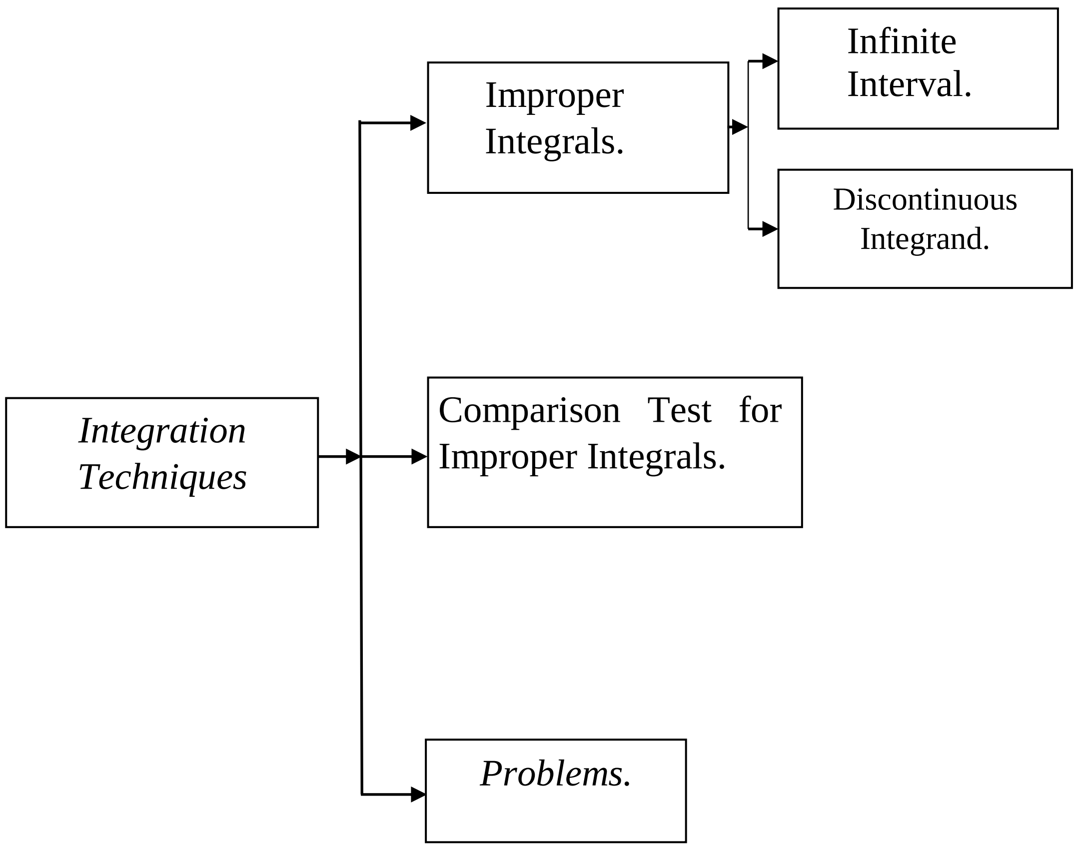
Lecture – Two

Integration Techniques



6- ***Improper Integrals*.**

A- Infinite Interval.

In this kind of integral one or both of the limits of integration are infinity. In these cases the interval of integration is said to be over an infinite interval.

Example 1 Evaluate the following integral.

$$\int_1^\infty \frac{1}{x^2}\,dx$$

Solution.

To see how we're going to do this integral let's think of this as an area problem. So instead of asking what the integral is, let's instead ask what the area under $f(x) = \frac{1}{x^2}$ on the interval $[1, \infty)$ is.

We still aren't able to do this, however, let's step back a little and instead ask what the area under $f(x)$ is on the interval $[1, t]$ where $t > 1$ and t is finite. This is a problem that we can do.

$$A_t = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^t = 1 - \frac{1}{t}$$

Now, we can get the area under $f(x)$ on $[1, \infty)$ simply by taking the limit of A_t as t goes to infinity.

$$A = \lim_{t\to\infty} A_t = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

This is then how we will do the integral itself.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx$$
$$= \lim_{t\to\infty}\left(-\frac{1}{x}\right)\Big|_1^t$$
$$= \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

Let's now get some definitions out of the way. We will call these integrals **convergent** if the associated limit exists and is a finite number (*i.e.* it's not plus or minus infinity) and **divergent** if the associated limit either doesn't exist or is (plus or minus) infinity.

Let's now formalize up the method for dealing with infinite intervals. There are essentially three cases that we'll need to look at.

1. If $\int_a^t f(x)\,dx$ exists for every $t > a$ then,

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided the limit exists and is finite.

2. If $\int_{t}^{b} f(x)\,dx$ exists for every $t < b$ then,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t \to -\infty} \int_{t}^{b} f(x)\,dx$$

provided the limits exists and is finite.

3. If $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$ are both convergent then,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

Example 2 Determine if the following integral is convergent or divergent and if it's convergent find its value.

$$\int_{1}^{\infty} \frac{1}{x}\,dx$$

Solution

So, the first thing we do is convert the integral to a limit.

$$\int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x}\,dx$$

Now, do the integral and the limit.

$$\begin{aligned} \int_{1}^{\infty} \frac{1}{x}\,dx &= \lim_{t \to \infty} \ln(x)\Big|_{1}^{t} \\ &= \lim_{t \to \infty} \left(\ln(t) - \ln 1\right) \\ &= \infty \end{aligned}$$

So, the limit is infinite and so the integral is divergent.

Fact

If $a > 0$ then

$$\int_{a}^{\infty} \frac{1}{x^{p}}\,dx$$

is convergent if $p > 1$ and divergent if $p \le 1$.

Example 3 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx$$

Solution

There really isn't much to do with these problems once you know how to do them. We'll convert the integral to a limit/integral pair, evaluate the integral and then the limit.

$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to-\infty}\int_{t}^{0} \frac{1}{\sqrt{3-x}}\,dx \\ &= \lim_{t\to-\infty} -2\sqrt{3-x}\,\Big|_{t}^{0} \\ &= \lim_{t\to-\infty}\left(-2\sqrt{3}+2\sqrt{3-t}\right) \\ &= -2\sqrt{3}+\infty \\ &= \infty\end{aligned}$$

So, the limit is infinite and so this integral is divergent.

Example 4 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx$$

Solution

In this case we've got infinities in both limits and so we'll need to split the integral up into two separate integrals. We can split the integral up at any point, so let's choose $a = 0$ since this will be a convenient point for the evaluation process. The integral is then,

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx = \int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx + \int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx$$

We've now got to look at each of the individual limits.

$$\begin{aligned}\int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx &= \lim_{t\to-\infty}\int_{t}^{0} x\mathbf{e}^{-x^2}\,dx \\ &= \lim_{t\to-\infty}\left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_{t}^{0} \\ &= \lim_{t\to-\infty}\left(-\frac{1}{2}+\frac{1}{2}\mathbf{e}^{-t^2}\right) \\ &= -\frac{1}{2}\end{aligned}$$

So, the first integral is convergent. Note that this does NOT mean that the second integral will also be convergent. So, let's take a look at that one.

$$\int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx = \lim_{t\to\infty}\int_{0}^{t} x\mathbf{e}^{-x^2}\,dx$$

$$
\begin{aligned}
&= \lim_{t\to\infty}\left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\Big|_0^t \\
&= \lim_{t\to\infty}\left(-\frac{1}{2}\mathbf{e}^{-t^2}+\frac{1}{2}\right) \\
&= \frac{1}{2}
\end{aligned}
$$

This integral is convergent and so since they are both convergent the integral we were actually asked to deal with is also convergent and its value is,

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx = \int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx + \int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx = -\frac{1}{2}+\frac{1}{2}=0$$

Example 5 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^{\infty}\sin x\,dx$$

Solution

First convert to a limit.

$$
\begin{aligned}
\int_{-2}^{\infty}\sin x\,dx &= \lim_{t\to\infty}\int_{-2}^{t}\sin x\,dx \\
&= \lim_{t\to\infty}\left(-\cos x\right)\Big|_{-2}^{t} \\
&= \lim_{t\to\infty}\left(\cos 2-\cos t\right)
\end{aligned}
$$

This limit doesn't exist and so the integral is divergent.

B- <u>Discontinuous Integrand.</u>

We now need to look at the second type of improper integrals that we'll be looking at in this section. These are integrals that have discontinuous integrands. The process here is basically the same with one subtle difference. Here are the general cases that we'll look at for these integrals.

1. If $f(x)$ is continuous on the interval $[a,b)$ and not continuous at $x=b$ then,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

provided the limit exists and is finite. Note as well that we do need to use a left hand limit here since the interval of integration is entirely on the left side of the upper limit.

2. If $f(x)$ is continuous on the interval $(a,b]$ and not continuous at $x=a$ then,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

provided the limit exists and is finite. In this case we need to use a right hand limit here since the interval of integration is entirely on the right side of the lower limit.

3. If $f(x)$ is not continuous at $x = c$ where $a < c < b$ and $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent then,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

As with the infinite interval case this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

4. If $f(x)$ is not continuous at $x = a$ and $x = b$ and if $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent then,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Where c is any number. Again, this requires BOTH of the integrals to be convergent in order for this integral to also be convergent.

Example 6 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$$

Solution
The problem point is the upper limit so we are in the first case above.

$$\begin{aligned}
\int_0^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to 3^-} \int_0^t \frac{1}{\sqrt{3-x}}\,dx \\
&= \lim_{t\to 3^-} \left(-2\sqrt{3-x}\right)\Big|_0^t \\
&= \lim_{t\to 3^-} \left(2\sqrt{3} - 2\sqrt{3-t}\right) \\
&= 2\sqrt{3}
\end{aligned}$$

The limit exists and is finite and so the integral converges and the integral's value is $2\sqrt{3}$.

Example 7 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^3 \frac{1}{x^3}\,dx$$

Solution
This integrand is not continuous at $x = 0$ and so we'll need to split the integral up at that point.

$$\int_{-2}^3 \frac{1}{x^3}\,dx = \int_{-2}^0 \frac{1}{x^3}\,dx + \int_0^3 \frac{1}{x^3}\,dx$$

Now we need to look at each of these integrals and see if they are convergent.

$$\begin{aligned}\int_{-2}^{0} \frac{1}{x^3}\,dx &= \lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x^3}\,dx \\ &= \lim_{t\to 0^-} \left(-\frac{1}{2x^2} \right)\Bigg|_{-2}^{t} \\ &= \lim_{t\to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{8} \right) \\ &= -\infty\end{aligned}$$

At this point we're done. One of the integrals is divergent that means the integral that we were asked to look at is divergent. We don't even need to bother with the second integral.

Example 8 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{\infty} \frac{1}{x^2}\,dx$$

Solution

This is an integral over an infinite interval that also contains a discontinuous integrand. To do this integral we'll need to split it up into two integrals. We can split it up anywhere, but pick a value that will be convenient for evaluation purposes.

$$\int_{0}^{\infty} \frac{1}{x^2}\,dx = \int_{0}^{1} \frac{1}{x^2}\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx$$

In order for the integral in the example to be convergent we will need BOTH of these to be convergent. If one or both are divergent then the whole integral will also be divergent.

We know that the second integral is convergent by the fact given in the infinite interval portion above. So, all we need to do is check the first integral.

$$\begin{aligned}\int_{0}^{1} \frac{1}{x^2}\,dx &= \lim_{t\to 0^+} \int_{t}^{1} \frac{1}{x^2}\,dx \\ &= \lim_{t\to 0^+} \left(-\frac{1}{x} \right)\Bigg|_{t}^{1} \\ &= \lim_{t\to 0^+} \left(-1 + \frac{1}{t} \right) \\ &= \infty\end{aligned}$$

7- ***Comparison Test for Improper Integrals***.

Now that we've seen how to actually compute improper integrals we need to address one more topic about them. Often we aren't concerned with the actual value of these integrals. Instead we might only be interested in whether the integral is convergent or divergent. Also, there will be some integrals that we simply won't be able to integrate and yet we would still like to know if they converge or diverge.

To deal with this we've got a test for convergence or divergence that we can use to help us answer the question of convergence for an improper integral.

Comparison Test

If $f(x) \geq g(x) \geq 0$ on the interval $[a, \infty)$ then,

1. If $\int_a^\infty f(x)\,dx$ converges then so does $\int_a^\infty g(x)\,dx$.
2. If $\int_a^\infty g(x)\,dx$ diverges then so does $\int_a^\infty f(x)\,dx$.

Note that if you think in terms of area the Comparison Test makes a lot of sense. If $f(x)$ is larger than $g(x)$ then the area under $f(x)$ must also be larger than the area under $g(x)$.

So, if the area under the larger function is finite (*i.e.* $\int_a^\infty f(x)\,dx$ converges) then the area under the smaller function must also be finite (*i.e.* $\int_a^\infty g(x)\,dx$ converges). Likewise, if the area under the smaller function is infinite (*i.e.* $\int_a^\infty g(x)\,dx$ diverges) then the area under the larger function must also be infinite (*i.e.* $\int_a^\infty f(x)\,dx$ diverges).

Example 1 Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$$

Solution.

So, it seems like it would be nice to have some idea as to whether the integral converges or diverges ahead of time so we will know whether we will need to look for a larger (and convergent) function or a smaller (and divergent) function.

Therefore, it seems likely that the denominator will determine the convergence/divergence of this integral and we know that

$$\int_2^\infty \frac{1}{x^2}\,dx$$

converges since $p = 2 > 1$ by the fact in the previous section. So let's guess that this integral will converge.

So we now know that we need to find a function that is larger than

$$\frac{\cos^2 x}{x^2}$$

and also converges. Making a fraction larger is actually a fairly simple process. We can either make the numerator larger or we can make the denominator smaller. In this case we can't do a lot about the denominator. However we can use the fact that $0 \le \cos^2 x \le 1$ to make the numerator larger (*i.e.* we'll replace the cosine with something we know to be larger, namely 1). So,

$$\frac{\cos^2 x}{x^2} \le \frac{1}{x^2}$$

Now, as we've already noted

$$\int_2^\infty \frac{1}{x^2}\,dx$$

converges and so by the Comparison Test we know that

$$\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$$

must also converge.

Example 2 Determine if the following integral is convergent or divergent.

$$\int_3^\infty \frac{1}{x + \mathbf{e}^x}\,dx$$

Solution

Let's first take a guess about the convergence of this integral. As noted after the fact in the last section about

The question then is which one to drop? Let's first drop the exponential. Doing this gives,

$$\frac{1}{x + \mathbf{e}^x} < \frac{1}{x}$$

This is a problem however, since

$$\int_3^\infty \frac{1}{x}\,dx$$

diverges by the fact. We've got a larger function that is divergent. This doesn't say anything about the smaller function. Therefore, we chose the wrong one to drop.

Let's try it again and this time let's drop the x.

$$\frac{1}{x + \mathbf{e}^x} < \frac{1}{\mathbf{e}^x} = \mathbf{e}^{-x}$$

Also,

$$\int_3^\infty \mathbf{e}^{-x}\,dx = \lim_{t\to\infty}\int_3^t \mathbf{e}^{-x}\,dx$$
$$= \lim_{t\to\infty}\left(-\mathbf{e}^{-t} + \mathbf{e}^{-3}\right)$$
$$= \mathbf{e}^{-3}$$

So, $\int_3^\infty \mathbf{e}^{-x}\,dx$ is convergent. Therefore, by the Comparison test

$$\int_3^\infty \frac{1}{x+\mathbf{e}^x}\,dx$$

is also convergent.

Example 3 Determine if the following integral is convergent or divergent.

$$\int_3^\infty \frac{1}{x-\mathbf{e}^{-x}}\,dx$$

Solution

This is where the second change will come into play. As before we know that both x and the exponential are positive. However, this time since we are subtracting the exponential from the x if we were to drop the exponential the denominator will become larger and so the fraction will become smaller. In other words,

$$\frac{1}{x-\mathbf{e}^{-x}} > \frac{1}{x}$$

and we know that

$$\int_3^\infty \frac{1}{x}\,dx$$

diverges and so by the Comparison Test we know that

$$\int_3^\infty \frac{1}{x-\mathbf{e}^{-x}}\,dx$$

must also diverge.

Example 4 Determine if the following integral is convergent or divergent.

$$\int_1^\infty \frac{1+3\sin^4(2x)}{\sqrt{x}}\,dx$$

Solution

Therefore, since the exponent on the denominator is less than 1 we can guess that the integral will probably diverge. We will need a smaller function that also diverges.

We know that $0 \le \sin^4(2x) \le 1$. In particular, this term is positive and so if we drop it from the numerator the numerator will get smaller. This gives,

$$\frac{1+3\sin^4(2x)}{\sqrt{x}} > \frac{1}{\sqrt{x}}$$

and

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

diverges so by the Comparison Test

$$\int_1^\infty \frac{1+3\sin^4(2x)}{\sqrt{x}}\,dx$$

also diverges.

8- Problems.

Improper Integrals

Determine if each of the following integrals converge or diverge. If the integral converges determine its value.

1. $\int_0^{\infty} (1+2x)\mathbf{e}^{-x}\,dx$

2. $\int_{-\infty}^{0} (1+2x)\mathbf{e}^{-x}\,dx$

3. $\int_{-5}^{1} \frac{1}{10+2z}\,dz$

4. $\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2-4}}\,dw$

5. $\int_{-\infty}^{1} \sqrt{6-y}\,dy$

6. $\int_{2}^{\infty} \frac{9}{(1-3z)^4}\,dz$

7. $\int_{0}^{4} \frac{x}{x^2-9}\,dx$

8. $\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw$

9. $\int_{1}^{4} \frac{1}{x^2+x-6}\,dx$

10. $\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$

Comparison Test for Improper Integrals

Use the Comparison Test to determine if the following integrals converge

1. $\int_{1}^{\infty} \frac{1}{x^3+1}\,dx$

2. $\int_{3}^{\infty} \frac{z^2}{z^3-1}\,dz$

3. $\int_{4}^{\infty} \frac{\mathbf{e}^{-y}}{y}\,dy$

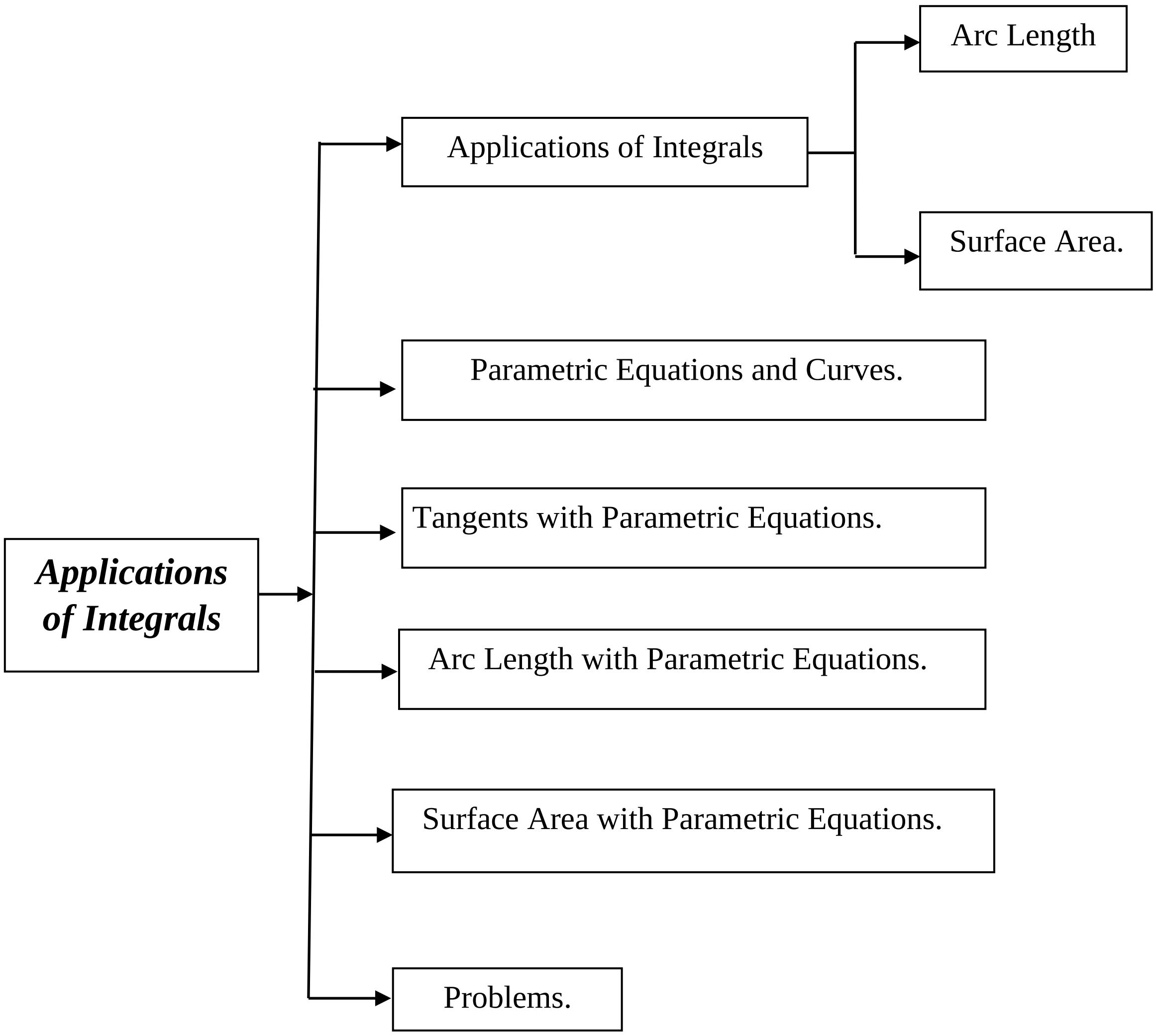
4. $\int_{1}^{\infty} \frac{z-1}{z^4+2z^2}\,dz$

By

Ms.C. Yasir R. Al-hamdany

Lecture – Three

Applications of Integrals

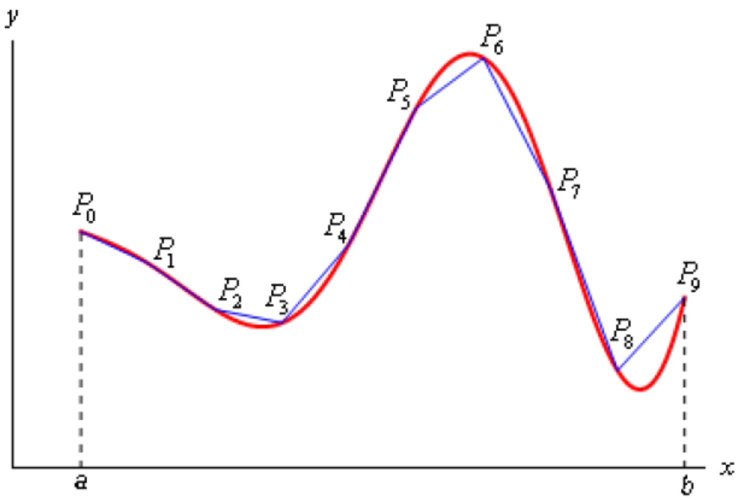


*1- **Applications of Integrals.***

A- Arc Length.

We want to determine the length of the continuous function $y = f(x)$ on the interval $[a,b]$. We'll also need to assume that the derivative is continuous on $[a,b]$.

Initially we'll need to estimate the length of the curve. We'll do this by dividing the interval up into n equal subintervals each of width Δx and we'll denote the point on the curve at each point by P_i. We can then approximate the curve by a series of straight lines connecting the points. Here is a sketch of this situation for $n = 9$.



Now denote the length of each of these line segments by $|P_{i-1}\ P_i|$ and the length of the curve will then be approximately,

$$L \approx \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

and we can get the exact length by taking n larger and larger. In other words, the exact length will be,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

Now, let's get a better grasp on the length of each of these line segments. First, on each segment let's define $\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$. We can then compute directly the length of the line segments as follows.

$$|P_{i-1}\ P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

By the Mean Value Theorem we know that on the interval $[x_{i-1}, x_i]$ there is a point x_i^* so that,

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$
$$\Delta y_i = f'(x_i^*)\,\Delta x$$

Therefore, the length can now be written as,

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + \left[f'(x_i^*)\right]^2 \Delta x^2} \\ &= \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$
$$\Delta y_i = f'(x_i^*)\,\Delta x$$

Therefore, the length can now be written as,

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + \left[f'(x_i^*)\right]^2 \Delta x^2} \\ &= \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

The exact length of the curve is then,

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

The exact length of the curve is then,

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

However, using the definition of the definite integral, this is nothing more than,

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

A slightly more convenient notation (in my opinion anyway) is the following.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

In a similar fashion we can also derive a formula for $x = h(y)$ on $[c, d]$. This formula is,

$$L = \int_c^d \sqrt{1 + [h'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Arc Length Formula(s)

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

Example 1 Determine the length of $y = \ln(\sec x)$ between $0 \le x \le \frac{\pi}{4}$.

Solution

In this case we'll need to use the first ds since the function is in the form $y = f(x)$. So, let's get the derivative out of the way.

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x \qquad\qquad \left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

Let's also get the root out of the way since there is often simplification that can be done and there's no reason to do that inside the integral.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 x} = \sqrt{\sec^2 x} = |\sec x| = \sec x$$

Note that we could drop the absolute value bars here since secant is positive in the range given.

The arc length is then,

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sec x\, dx \\ &= \ln|\sec x + \tan x| \Big|_0^{\frac{\pi}{4}} \\ &= \ln\left(\sqrt{2} + 1\right) \end{aligned}$$

Example 2 Determine the length of $x=\frac{2}{3}(y-1)^{\frac{3}{2}}$ between $1\le y\le 4$.

Solution

Let's compute the derivative and the root.

$$\frac{dx}{dy}=(y-1)^{\frac{1}{2}} \qquad \Rightarrow \qquad \sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+y-1}=\sqrt{y}$$

As you can see keeping the function in the form $x=h(y)$ is going to lead to a very easy integral. To see what would happen if we tried to work with the function in the form $y=f(x)$ see the next example.

Let's get the length.

$$\begin{aligned} L &= \int_1^4 \sqrt{y}\, dy \\ &= \frac{2}{3}y^{\frac{3}{2}}\Big|_1^4 \\ &= \frac{14}{3} \end{aligned}$$

Example 3 Redo the previous example using the function in the form $y=f(x)$ instead.

Solution

In this case the function and its derivative would be,

$$y=\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1 \qquad\qquad \frac{dy}{dx}=\left(\frac{3x}{2}\right)^{-\frac{1}{3}}$$

The root in the arc length formula would then be.

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\sqrt{1+\frac{1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}}=\sqrt{\frac{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}}=\frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}}$$

All the simplification work above was just to put the root into a form that will allow us to do the integral.

Now, before we write down the integral we'll also need to determine the limits. This particular ds requires x limits of integration and we've got y limits. They are easy enough to get however. Since we know x as a function of y all we need to do is plug in the original y limits of integration and get the x limits of integration. Doing this gives,

$$0\le x\le \frac{2}{3}(3)^{\frac{3}{2}}$$

Not easy limits to deal with, but there they are.

Let's now write down the integral that will give the length.

$$L = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}}\,dx$$

That's a really unpleasant looking integral. It can be evaluated however using the following substitution.

$$u = \left(\frac{3x}{2}\right)^{\frac{2}{3}}+1 \qquad\qquad du = \left(\frac{3x}{2}\right)^{-\frac{1}{3}} dx$$

$$x = 0 \qquad \Rightarrow \qquad u = 1$$

$$x = \frac{2}{3}(3)^{\frac{3}{2}} \qquad \Rightarrow \qquad u = 4$$

Using this substitution the integral becomes,

$$\begin{aligned} L &= \int_1^4 \sqrt{u}\,du \\ &= \left.\frac{2}{3}u^{\frac{3}{2}}\right|_1^4 \\ &= \frac{14}{3} \end{aligned}$$

So, we got the same answer as in the previous example. Although that shouldn't really be all that surprising since we were dealing with the same curve.

Example 4 Determine the length of $x = \frac{1}{2}y^2$ for $0 \le x \le \frac{1}{2}$. Assume that y is positive.

Solution

We'll use the second *ds* for this one as the function is already in the correct form for that one. Also, the other *ds* would again lead to a particularly difficult integral. The derivative and root will then be,

$$\frac{dx}{dy} = y \qquad \Rightarrow \qquad \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+y^2}$$

Before writing down the length notice that we were given x limits and we will need y limits for this *ds*. With the assumption that y is positive these are easy enough to get. All we need to do is plug x into our equation and solve for y. Doing this gives,

$$0 \le y \le 1$$

The integral for the arc length is then,

$$L = \int_0^1 \sqrt{1+y^2}\,dy$$

This integral will require the following trig substitution.

$$y = \tan\theta \qquad\qquad dy = \sec^2\theta\,d\theta$$

$$y = 0 \quad \Rightarrow \quad 0 = \tan\theta \quad \Rightarrow \quad \theta = 0$$

$$y = 1 \quad \Rightarrow \quad 1 = \tan\theta \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\sqrt{1+y^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$$

The length is then,

$$L = \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right)$$

B- ***Surface Area.***

The surface area of a frustum is given by,

$$A = 2\pi r l$$

where,

$$r = \frac{1}{2}(r_1 + r_2) \qquad r_1 = \text{radius of right end}$$

$$r_2 = \text{radius of left end}$$

and l is the length of the slant of the frustum.

For the frustum on the interval $[x_{i-1}, x_i]$ we have,

$$r_1 = f(x_i)$$

$$r_2 = f(x_{i-1})$$

$$l = |P_{i-1}\ P_i| \quad (\text{length of the line segment connecting } P_i \text{ and } P_{i-1})$$

We know from the previous section that,

$$|P_{i-1}\ P_i| = \sqrt{1+\left[f'(x_i^*)\right]^2}\ \Delta x \quad \text{where } x_i^* \text{ is some point in } [x_{i-1}, x_i]$$

Before writing down the formula for the surface area we are going to assume that Δx is “small” and since $f(x)$ is continuous we can then assume that,

$$f(x_i) \approx f(x_i^*) \qquad \text{and} \qquad f(x_{i-1}) \approx f(x_i^*)$$

So, the surface area of the frustum on the interval $[x_{i-1}, x_i]$ is approximately,

$$A_i = 2\pi\left(\frac{f(x_i)+f(x_{i-1})}{2}\right)|P_{i-1}\ P_i|$$
$$\approx 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\ \ \Delta x$$

The surface area of the whole solid is then approximately,

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\ \ \Delta x$$

and we can get the exact surface area by taking the limit as n goes to infinity.

$$S = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\ \ \Delta x$$
$$= \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

If we wanted to we could also derive a similar formula for rotating $x = h(y)$ on $[c,d]$ about the y-axis. This would give the following formula.

$$S = \int_c^d 2\pi\, h(y)\sqrt{1+\left[h'(y)\right]^2}\,dy$$

Surface Area Formulas

$$S = \int 2\pi y\,ds \qquad \text{rotation about } x-\text{axis}$$
$$S = \int 2\pi x\,ds \qquad \text{rotation about } y-\text{axis}$$

where,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{if } y = f(x),\ a \le x \le b$$
$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{if } x = h(y),\ c \le y \le d$$

Example 1 Determine the surface area of the solid obtained by rotating $y = \sqrt{9-x^2}$, $-2 \le x \le 2$ about the x-axis.

Solution

The formula that we'll be using here is,

$$S = \int 2\pi y\,ds$$

Let's first get the derivative and the root taken care of.

$$\frac{dy}{dx} = \frac{1}{2}\left(9-x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{\left(9-x^2\right)^{\frac{1}{2}}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{9-x^2}} = \sqrt{\frac{9}{9-x^2}} = \frac{3}{\sqrt{9-x^2}}$$

Here's the integral for the surface area,

$$S = \int_{-2}^{2} 2\pi y \frac{3}{\sqrt{9-x^2}}\,dx$$

There is a problem however. The dx means that we shouldn't have any y's in the integral. So, before evaluating the integral we'll need to substitute in for y as well.

The surface area is then,

$$\begin{aligned} S &= \int_{-2}^{2} 2\pi\sqrt{9-x^2}\,\frac{3}{\sqrt{9-x^2}}\,dx \\ &= \int_{-2}^{2} 6\pi\,dx \\ &= 24\pi \end{aligned}$$

Example 2 Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}$, $1 \le y \le 2$ about the y-axis. Use both ds's to compute the surface area.

Solution

Note that we've been given the function set up for the first ds and limits that work for the second ds.

Solution 1

This solution will use the first ds listed above. We'll start with the derivative and root.

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{9x^{\frac{4}{3}}}} = \sqrt{\frac{9x^{\frac{4}{3}}+1}{9x^{\frac{4}{3}}}} = \frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}$$

We'll also need to get new limits. That isn't too bad however. All we need to do is plug in the given y's into our equation and solve to get that the range of x's is $1 \le x \le 8$. The integral for the surface area is then,

$$\begin{aligned} S &= \int_{1}^{8} 2\pi x \frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}\,dx \\ &= \frac{2\pi}{3}\int_{1}^{8} x^{\frac{1}{3}}\sqrt{9x^{\frac{4}{3}}+1}\,dx \end{aligned}$$

Using the substitution

$$u = 9x^{\frac{4}{3}} + 1 \qquad\qquad du = 12x^{\frac{1}{3}}\,dx$$

the integral becomes,

$$S = \frac{\pi}{18}\int_{10}^{145}\sqrt{u}\,du$$
$$= \frac{\pi}{27}u^{\frac{3}{2}}\Big|_{10}^{145}$$
$$= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48$$

Solution 2
This time we'll use the second ds. So, we'll first need to solve the equation for x. We'll also go ahead and get the derivative and root while we're at it.

$$x = y^3 \qquad\qquad \frac{dx}{dy} = 3y^2$$
$$\sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+9y^4}$$

The surface area is then,

$$S = \int_1^2 2\pi x\sqrt{1+9y^4}\,dy$$

We used the original y limits this time because we picked up a dy from the ds. Also note that the presence of the dy means that this time, unlike the first solution, we'll need to substitute in for the x. Doing that gives,

$$S = \int_1^2 2\pi y^3\sqrt{1+9y^4}\,dy \qquad\qquad u = 1+9y^4$$
$$= \frac{\pi}{18}\int_{10}^{145}\sqrt{u}\,du$$
$$= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48$$

Note that after the substitution the integral was identical to the first solution and so the work was skipped.

2- ***Parametric Equations and Curves***.

To this point (in both Calculus I and Calculus II) we've looked almost exclusively at functions in the form $y = f(x)$ or $x = h(y)$ and almost all of the formulas that we've developed require that functions be in one of these two forms. The problem is that not all curves or equations that we'd like to look at fall easily into this form.

Take, for example, a circle. It is easy enough to write down the equation of a circle centered at the origin with radius r.

$$x^2 + y^2 = r^2$$

However, we will never be able to write the equation of a circle down as a single equation in either of the forms above. Sure we can solve for x or y as the following two formulas show

$$y = \pm\sqrt{r^2 - x^2} \qquad\qquad x = \pm\sqrt{r^2 - y^2}$$

but there are in fact two functions in each of these. Each formula gives a portion of the circle.

$$y = \sqrt{r^2 - x^2} \quad (\text{top}) \qquad\qquad x = \sqrt{r^2 - y^2} \quad (\text{right side})$$

$$y = -\sqrt{r^2 - x^2} \quad (\text{bottom}) \qquad\qquad x = -\sqrt{r^2 - y^2} \quad (\text{left side})$$

There are also a great many curves out there that we can't even write down as a single equation in terms of only x and y. So, to deal with some of these problems we introduce **parametric equations**. Instead of defining y in terms of x ($y = f(x)$) or x in terms of y ($x = h(y)$) we define both x and y in terms of a third variable called a parameter as follows,

$$x = f(t) \qquad\qquad y = g(t)$$

Each value of t defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting t be all possible values is the graph of the parametric equations and is called the **parametric curve**.

Example 1 Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad\qquad y = 2t - 1$$

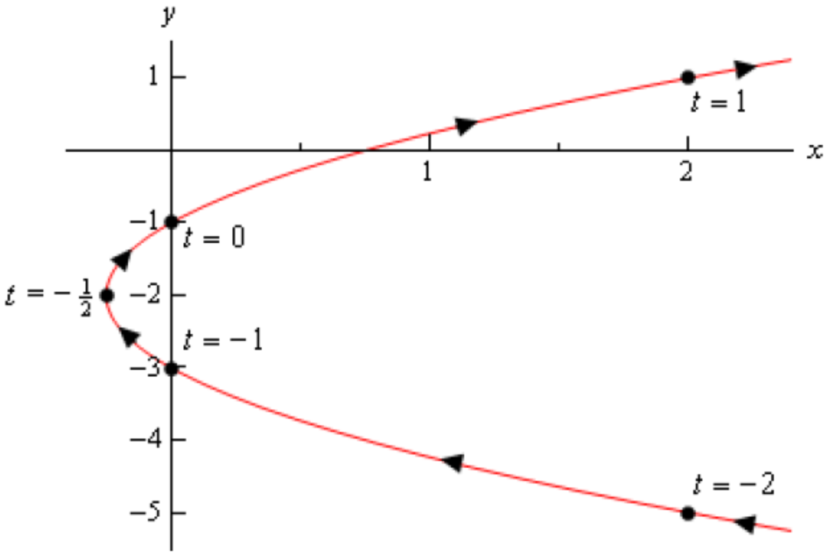
Solution

At this point our only option for sketching a parametric curve is to pick values of t, plug them into the parametric equations and then plot the points. So, let's plug in some t's.

t	x	y
-2	2	-5
-1	0	-3
$-\frac{1}{2}$	$-\frac{1}{4}$	-2
0	0	-1
1	2	1

We have one more idea to discuss before we actually sketch the curve. Parametric curves have a **direction of motion**. The direction of motion is given by increasing t. So, when plotting parametric curves we also include arrows that show the direction of motion.

Here is the sketch of this parametric curve.

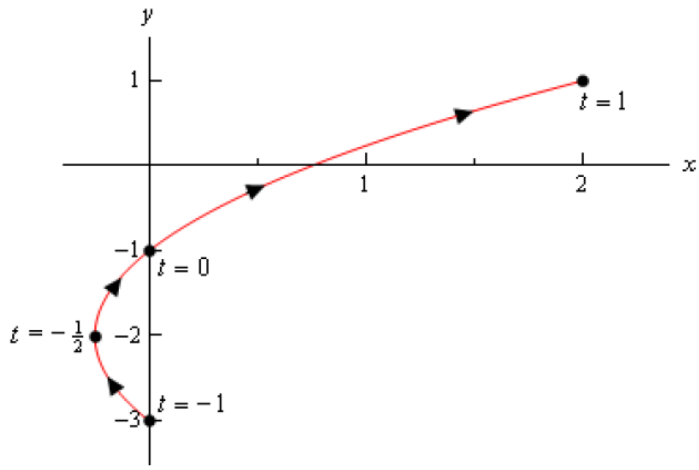


Example 2 Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \qquad -1 \le t \le 1$$

Solution

Note that the only difference here is the presence of the limits on t. All these limits do is tell us that we can't take any value of t outside of this range. Therefore, the parametric curve will only be a portion of the curve above. Here is the parametric curve for this example.



Example 3 Sketch the parametric curve for the following set of parametric equations. Clearly indicate direction of motion.

$$x = 5\cos t \qquad y = 2\sin t \qquad 0 \le t \le 2\pi$$

Solution

An alternate method that we could have used here was to solve the two parametric equations for sine and cosine as follows,

$$\cos t = \frac{x}{5} \qquad\qquad \sin t = \frac{y}{2}$$

Then, recall the trig identity we used above and these new equation we get,

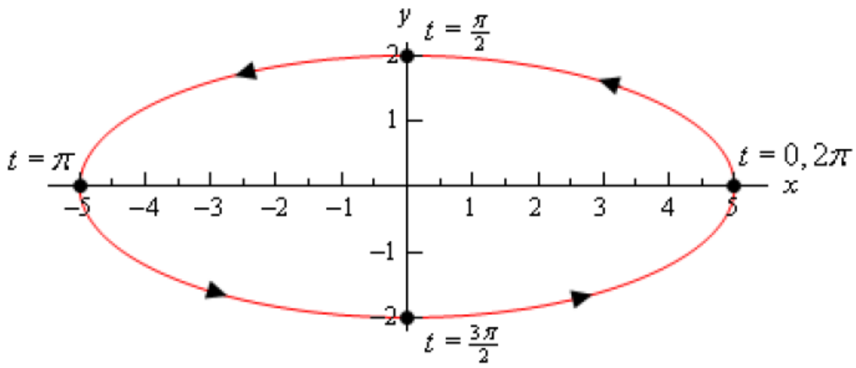
$$1 = \cos^2 t + \sin^2 t = \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = \frac{x^2}{25} + \frac{y^2}{4}$$

So, here is a table of values for this set of parametric equations.

t	x	y
0	5	0
$\frac{\pi}{2}$	0	2
π	-5	0
$\frac{3\pi}{2}$	0	-2
2π	5	0

It looks like we are moving in a counter-clockwise direction about the ellipse and it also looks like we'll make exactly one complete trace of the ellipse in the range given.

Here is a sketch of the parametric curve.



Example 4

The path of a particle is given by the following set of parametric equations.

$$x = 3\cos(2t) \qquad\qquad y = 1 + \cos^2(2t)$$

Completely describe the path of this particle. Do this by sketching the path, determining limits on x and y and giving a range of t's for which the path will be traced out exactly once (provide it traces out more than once of course).

Solution

Eliminating the parameter this time will be a little different. We only have cosines this time and we'll use that to our advantage. We can solve the x equation for cosine and plug that into the equation for y. This gives,

$$\cos(2t) = \frac{x}{3} \qquad\qquad y = 1 + \left(\frac{x}{3}\right)^2 = 1 + \frac{x^2}{9}$$

This time we've got a parabola that opens upward. We also have the following limits on x and y.

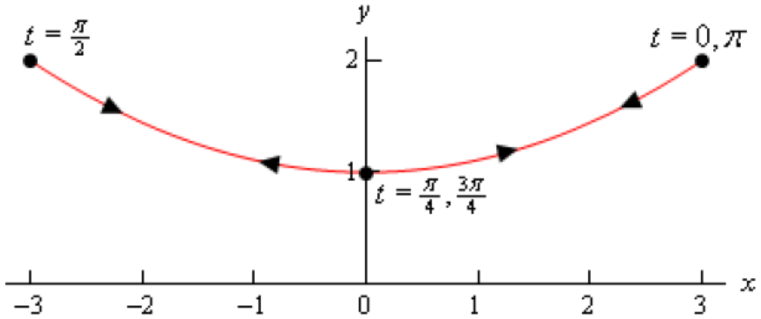
$$-1 \le \cos(2t) \le 1 \qquad -3 \le 3\cos(2t) \le 3 \qquad -3 \le x \le 3$$

$$0 \le \cos^2(2t) \le 1 \qquad 1 \le 1 + \cos^2(2t) \le 2 \qquad 1 \le y \le 2$$

So, again we only trace out a portion of the curve. Here's a set of evaluations so we can determine a range of t's for one trace of the curve.

t	x	y
0	3	2
$\frac{\pi}{4}$	0	1
$\frac{\pi}{2}$	-3	2
$\frac{3\pi}{4}$	0	1
π	3	2

So, it looks like the particle, again, will continuously trace out this portion of the curve and will make one trace in the range $0 \le t \le \frac{\pi}{2}$. Here is a sketch of the particle's path with a few value of t on it.



3- ***Tangents with Parametric Equations.***

In this section we want to find the tangent lines to the parametric equations given by,

$$x = f(t) \qquad\qquad y = g(t)$$

To do this let's first recall how to find the tangent line to $y = F(x)$ at $x = a$. Here the tangent line is given by,

$$y = F(a) + m(x - a), \quad \text{where} \quad m = \left.\frac{dy}{dx}\right|_{x=a} = F'(a)$$

Now, notice that if we could figure out how to get the derivative $\frac{dy}{dx}$ from the parametric equations we could simply reuse this formula since we will be able to use the parametric equations to find the x and y coordinates of the point.

So, just for a second let's suppose that we were able to eliminate the parameter from the parametric form and write the parametric equations in the form $y = F(x)$. Now, plug the parametric equations in for x and y. Yes, it seems silly to eliminate the parameter, then immediately put it back in, but it's what we need to do in order to get our hands on the derivative. Doing this gives,

$$g(t) = F(f(t))$$

Now, differentiate with respect to t and notice that we'll need to use the Chain Rule on the right hand side.

$$g'(t) = F'(f(t))\, f'(t)$$

Let's do another change in notation. We need to be careful with our derivatives here. Derivatives of the lower case function are with respect to t while derivatives of upper case functions are with respect to x. So, to make sure that we keep this straight let's rewrite things as follows.

$$\frac{dy}{dt} = F'(x)\frac{dx}{dt}$$

At this point we should remind ourselves just what we are after. We needed a formula for $\frac{dy}{dx}$ or $F'(x)$ that is in terms of the parametric formulas. Notice however that we can get that from the above equation.

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \text{provided} \quad \frac{dx}{dt} \neq 0}$$

Derivative for Parametric Equations

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}, \quad \text{provided} \quad \frac{dy}{dt} \neq 0$$

Example 1 Find the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \qquad\qquad y = t^2$$

at (0,4).

Solution

The first thing that we should do is find the derivative so we can get the slope of the tangent line.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

At this point we've got a small problem. The derivative is in terms of t and all we've got is an x-y coordinate pair. The next step then is to determine the value(s) of t which will give this point. We find these by plugging the x and y values into the parametric equations and solving for t.

$$0 = t^5 - 4t^3 = t^3\left(t^2 - 4\right) \qquad \Rightarrow \qquad t = 0, \pm 2$$

$$4 = t^2 \qquad \Rightarrow \qquad t = \pm 2$$

$t = -2$

Since we already know the x and y-coordinates of the point all that we need to do is find the slope of the tangent line.

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$$

The tangent line (at $t = -2$) is then,

$$y = 4 - \frac{1}{8}x$$

$t = 2$

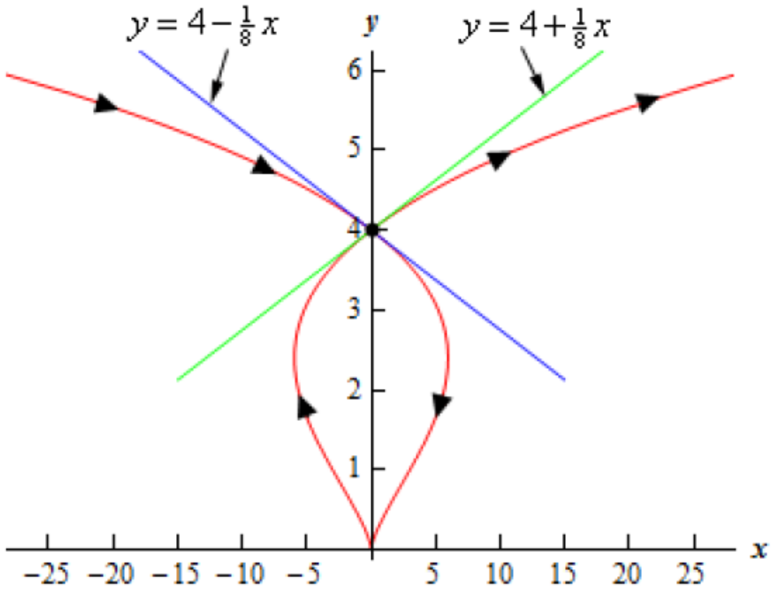
Again, all we need is the slope.

$$m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$$

The tangent line (at $t = 2$) is then,

$$y = 4 + \frac{1}{8}x$$

A quick graph of the parametric curve will explain what is going on here.



Horizontal Tangent for Parametric Equations

$$\frac{dy}{dt} = 0, \quad \text{provided} \quad \frac{dx}{dt} \neq 0$$

Vertical tangents will occur where the derivative is not defined and so we'll get vertical tangents at values of t for which we have,

Vertical Tangent for Parametric Equations

$$\frac{dx}{dt} = 0, \quad \text{provided} \quad \frac{dy}{dt} \neq 0$$

Example 2 Determine the x-y coordinates of the points where the following parametric equations will have horizontal or vertical tangents.

$$x = t^3 - 3t \qquad\qquad y = 3t^2 - 9$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = 3t^2 - 3 = 3\left(t^2 - 1\right) \qquad\qquad \frac{dy}{dt} = 6t$$

Horizontal Tangents

We'll have horizontal tangents where,

$$6t = 0 \qquad \Rightarrow \qquad t = 0$$

Now, this is the value of t which gives the horizontal tangents and we were asked to find the x-y coordinates of the point. To get these we just need to plug t into the parametric equations. Therefore, the only horizontal tangent will occur at the point (0,-9).

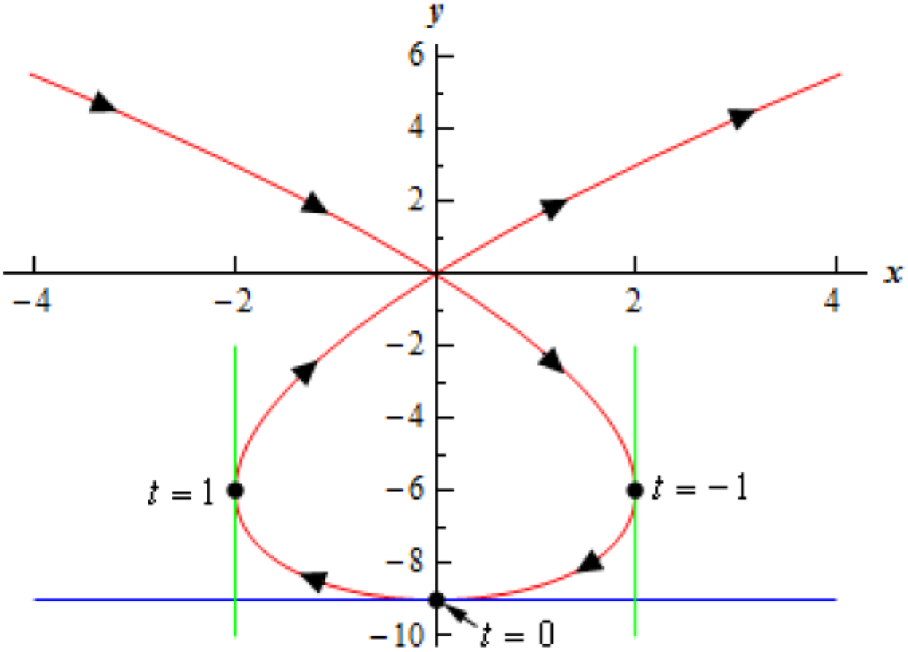
Vertical Tangents

In this case we need to solve,

$$3\left(t^2 - 1\right) = 0 \qquad \Rightarrow \qquad t = \pm 1$$

The two vertical tangents will occur at the points (2,-6) and (-2,-6).

For the sake of completeness and at least partial verification here is the sketch of the parametric curve.



4- <u>*Arc Length with Parametric Equations.*</u>

In this section we will look at the arc length of the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

We will also be assuming that the curve is traced out exactly once as t increases from α to β. We will also need to assume that the curve is traced out from left to right as t increases. This is equivalent to saying,

$$\frac{dx}{dt} \ge 0 \qquad \text{for } \alpha \le t \le \beta$$

To use this we'll also need to know that,

$$dx = f'(t)\,dt = \frac{dx}{dt}\,dt$$

The arc length formula then becomes,

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\,\frac{dx}{dt}\,dt = \int_{\alpha}^{\beta} \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\,\frac{dx}{dt}\,dt$$

$$L = \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; \frac{dx}{dt} dt$$

Now, making use of our assumption that the curve is being traced out from left to right we can drop the absolute value bars on the derivative which will allow us to cancel the two derivatives that are outside the square root and this gives,

Arc Length for Parametric Equations

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

Notice that we could have used the second formula for ds above if we had assumed instead that

$$\frac{dy}{dt} \geq 0 \qquad \text{for } \alpha \leq t \leq \beta$$

Example 1 Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(t) \qquad y = 3\cos(t) \qquad 0 \leq t \leq 2\pi$$

Solution

So, we can use the formula we derived above. We'll first need the following,

$$\frac{dx}{dt} = 3\cos(t) \qquad \frac{dy}{dt} = -3\sin(t)$$

The length is then,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)} \; dt \\ &= \int_0^{2\pi} 3\sqrt{\sin^2(t) + \cos^2(t)} \; dt \\ &= 3\int_0^{2\pi} dt \\ &= 6\pi \end{aligned}$$

Example 2 Use the arc length formula for the following parametric equations.

$$x = 3\sin(3t) \qquad y = 3\cos(3t) \qquad 0 \le t \le 2\pi$$

Solution

Notice that this is the identical circle that we had in the previous example and so the length is still 6π. However, for the range given we know it will trace out the curve three times instead once as required for the formula. Despite that restriction let's use the formula anyway and see what happens.

In this case the derivatives are,

$$\frac{dx}{dt} = 9\cos(3t) \qquad \frac{dy}{dt} = -9\sin(3t)$$

and the length formula gives,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{81\sin^2(t) + 81\cos^2(t)}\, dt \\ &= \int_0^{2\pi} 9\, dt \\ &= 18\pi \end{aligned}$$

The arc length formula can be summarized as,

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{if } x = f(t), y = g(t),\ \alpha \le t \le \beta$$

5- ***Surface Area with Parametric Equations.***

In this final section of looking at calculus applications with parametric equations we will take a look at determining the surface area of a region obtained by rotating a parametric curve about the x or y-axis.

We will rotate the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

about the x or y-axis. We are going to assume that the curve is traced out exactly once as t increases from α to β. At this point there actually isn't all that much to do. We know that the surface area can be found by using one of the following two formulas depending on the axis of rotation (recall the Surface Area section of the Applications of Integrals chapter).

$$S = \int 2\pi y\, ds \qquad \text{rotation about } x-\text{axis}$$

$$S = \int 2\pi x\, ds \qquad \text{rotation about } y-\text{axis}$$

All that we need is a formula for ds to use and from the previous section we have,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{if } x = f(t), y = g(t), \alpha \le t \le \beta$$

which is exactly what we need.

Example 1 Determine the surface area of the solid obtained by rotating the following parametric curve about the x-axis.

$$x = \cos^3\theta \qquad y = \sin^3\theta \qquad 0 \le \theta \le \frac{\pi}{2}$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = -3\cos^2\theta\sin\theta \qquad\qquad \frac{dy}{dt} = 3\sin^2\theta\cos\theta$$

Before plugging into the surface area formula let's get the ds out of the way.

$$\begin{aligned} ds &= \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}\, dt \\ &= 3\left|\cos\theta\sin\theta\right|\sqrt{\cos^2\theta + \sin^2\theta} \\ &= 3\cos\theta\sin\theta \end{aligned}$$

Notice that we could drop the absolute value bars since both sine and cosine are positive in this range of θ given.

Now let's get the surface area and don't forget to also plug in for the y.

$$\begin{aligned} S &= \int 2\pi y\, ds \\ &= 2\pi\int_0^{\frac{\pi}{2}} \sin^3\theta\left(3\cos\theta\sin\theta\right)\, d\theta \\ &= 6\pi\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\, d\theta \qquad\qquad u = \sin\theta \\ &= 6\pi\int_0^1 u^4\, du \\ &= \frac{6\pi}{5} \end{aligned}$$

6- Problems.

A- Arc Length.

1. Set up, but do not evaluate, an integral for the length of $y=\sqrt{x+2}$, $1 \le x \le 7$ using,

(a) $ds=\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$

(b) $ds=\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$

2. Set up, but do not evaluate, an integral for the length of $x=\cos(y)$, $0 \le x \le \frac{1}{2}$ using,

(a) $ds=\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$

(b) $ds=\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$

3. Determine the length of $y=7(6+x)^{\frac{3}{2}}$, $189 \le y \le 875$.

4. Determine the length of $x=4(3+y)^2$, $1 \le y \le 4$.

B- Surface Area.

1. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $x=\sqrt{y+5}$, $\sqrt{5} \le x \le 3$ about the y-axis using,

(a) $ds=\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$

(b) $ds=\sqrt{1+\left[\frac{dx}{dy}\right]^2}\,dy$

2. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y=\sin(2x)$, $0 \le x \le \frac{\pi}{8}$ about the x-axis using,

(a) $ds=\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$

(b) $ds = \sqrt{1+\left[\dfrac{dx}{dy}\right]^2}\, dy$

3. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y = x^3 + 4$, $1 \le x \le 5$ about the given axis. You can use either ds.

   (a) x-axis

   (b) y-axis

4. Find the surface area of the object obtained by rotating $y = 4 + 3x^2$, $1 \le x \le 2$ about the y-axis.

5. Find the surface area of the object obtained by rotating $y = \sin(2x)$, $0 \le x \le \frac{\pi}{8}$ about the x-axis.

C- Parametric Equations and Curves.

For problems 1 – 6 eliminate the parameter for the given set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on x and y.

1. $x = 4 - 2t \qquad y = 3 + 6t - 4t^2$

2. $x = 4 - 2t \qquad y = 3 + 6t - 4t^2 \qquad 0 \le t \le 3$

3. $x = \sqrt{t+1} \qquad y = \dfrac{1}{t+1} \qquad t > -1$

4. $x = 3\sin(t) \qquad y = -4\cos(t) \qquad 0 \le t \le 2\pi$

5. $x = 3\sin(2t) \qquad y = -4\cos(2t) \qquad 0 \le t \le 2\pi$

6. $x = 3\sin\left(\frac{1}{3}t\right) \qquad y = -4\cos\left(\frac{1}{3}t\right) \qquad 0 \le t \le 2\pi$

For problems 7 – 11 the path of a particle is given by the set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

(*i*) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

(*ii*) Limits on x and y.

(*iii*) A range of t's for a single trace of the parametric curve.

7. $x = 3 - 2\cos(3t) \qquad y = 1 + 4\sin(3t)$

8. $x = 4\sin\left(\frac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\frac{1}{4}t\right) \qquad -52\pi \le t \le 34\pi$

9. $x = \sqrt{4 + \cos\left(\frac{5}{2}t\right)} \qquad y = 1 + \frac{1}{3}\cos\left(\frac{5}{2}t\right) \qquad -48\pi \le t \le 2\pi$

10. $x = 2\mathbf{e}^{t} \qquad y = \cos\left(1 + \mathbf{e}^{3t}\right) \qquad 0 \le t \le \frac{3}{4}$

11. $x = \frac{1}{2}\mathbf{e}^{-3t} \qquad y = \mathbf{e}^{-6t} + 2\mathbf{e}^{-3t} - 8$

D- Tangents with Parametric Equations.

For problems 1 and 2 compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the given set of parametric equations.

1. $x = 4t^3 - t^2 + 7t \qquad y = t^4 - 6$

2. $x = \mathbf{e}^{-7t} + 2 \qquad y = 6\mathbf{e}^{2t} + \mathbf{e}^{-3t} - 4t$

For problems 3 and 4 find the equation of the tangent line(s) to the given set of parametric equations at the given point.

3. $x = 2\cos(3t) - 4\sin(3t) \qquad y = 3\tan(6t)$ at $t = \frac{\pi}{2}$

4. $x = t^2 - 2t - 11 \qquad y = t(t-4)^3 - 3t^2(t-4)^2 + 7$ at $(-3, 7)$

5. Find the values of t that will have horizontal or vertical tangent lines for the following set of parametric equations.

$$x = t^5 - 7t^4 - 3t^3 \qquad y = 2\cos(3t) + 4t$$

E- Area with Parametric Equations.

For problems 1 and 2 determine the area of the region below the parametric curve given by the set of parametric equations. For each problem you may assume that each curve traces out exactly once from right to left for the given range of t. For these problems you should only use the given parametric equations to determine the answer.

1. $x = 4t^3 - t^2 \qquad y = t^4 + 2t^2 \qquad 1 \le t \le 3$

2. $x = 3 - \cos^3(t) \qquad y = 4 + \sin(t) \qquad 0 \le t \le \pi$

F- Arc Length with Parametric Equations.

For problems 1 and 2 determine the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of t's.

1. $x = 8t^{\frac{3}{2}} \qquad y = 3 + (8-t)^{\frac{3}{2}} \qquad 0 \le t \le 4$

2. $x = 3t+1 \qquad y = 4-t^2 \qquad -2 \le t \le 0$

3. A particle travels along a path defined by the following set of parametric equations. Determine the total distance the particle travels and compare this to the length of the parametric curve itself.

$$x = 4\sin\left(\tfrac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\tfrac{1}{4}t\right) \qquad -52\pi \le t \le 34\pi$$

For problems 4 and 5 set up, but do not evaluate, an integral that gives the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of t's.

4. $x = 2+t^2 \qquad y = \mathbf{e}^t \sin(2t) \qquad 0 \le t \le 3$

5. $x = \cos^3(2t) \qquad y = \sin(1-t^2) \qquad -\tfrac{3}{2} \le t \le 0$

G- <u>Surface Area with Parametric Equations.</u>

For problems1 – 3 determine the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of t's.

1. Rotate $x = 3+2t \qquad y = 9-3t \qquad 1 \le t \le 4$ about the y-axis.

2. Rotate $x = 9+2t^2 \qquad y = 4t \qquad 0 \le t \le 2$ about the x-axis.

3. Rotate $x = 3\cos(\pi t) \qquad y = 5t+2 \qquad 0 \le t \le \tfrac{1}{2}$ about the y-axis.

For problems 4 and 5 set up, but do not evaluate, an integral that gives the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of t's.

4. Rotate $x = 1+\ln(5+t^2) \qquad y = 2t-2t^2 \qquad 0 \le t \le 2$ about the x-axis.

5. Rotate $x = 1+3t^2 \qquad y = \sin(2t)\cos\left(\tfrac{1}{4}t\right) \qquad 0 \le t \le \tfrac{1}{2}$ about the y-axis.

By

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Lecture – Four

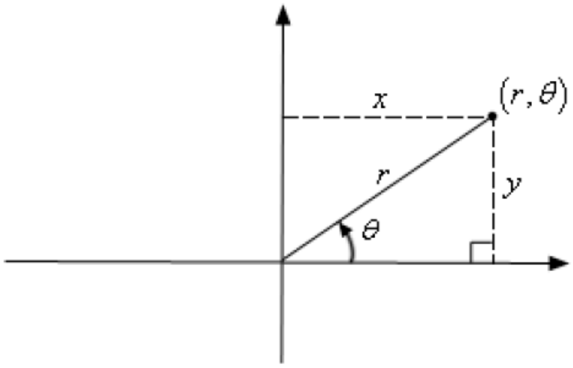
***Polar Coordinates* Technique**

Polar Coordinates Technique

- *Polar Coordinates*
- Common Polar Coordinate Graphs.
- *Tangents with Polar Coordinates.*
- Arc Length with Polar Coordinates.
- Area Polar Coordinates
- Problems.

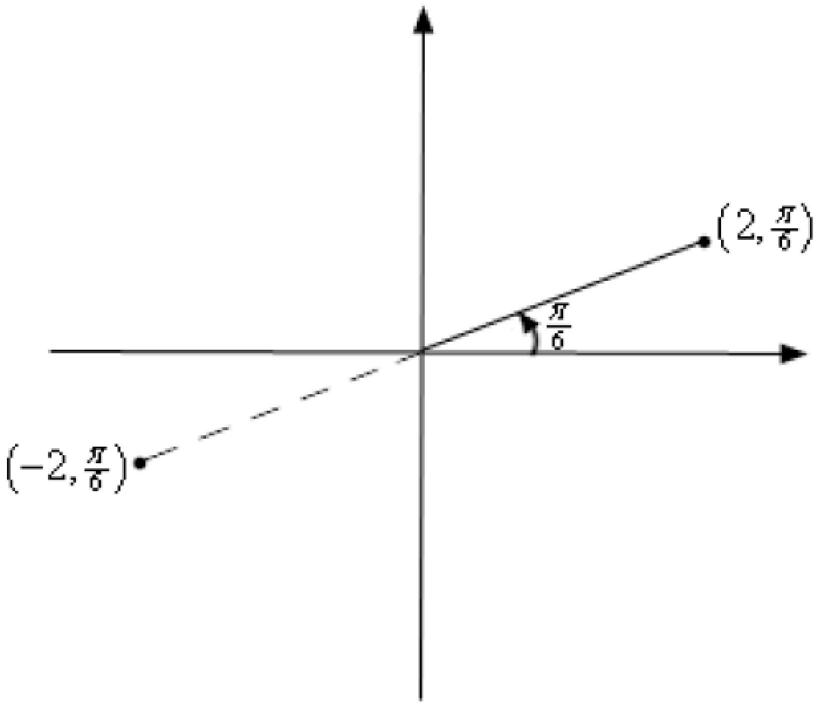
1- *Polar Coordinates*

This is not, however, the only way to define a point in two dimensional space. Instead of moving vertically and horizontally from the origin to get to the point we could instead go straight out of the origin until we hit the point and then determine the angle this line makes with the positive x-axis. We could then use the distance of the point from the origin and the amount we needed to rotate from the positive x-axis as the coordinates of the point. This is shown in the sketch below.



Coordinates in this form are called **polar coordinates**.

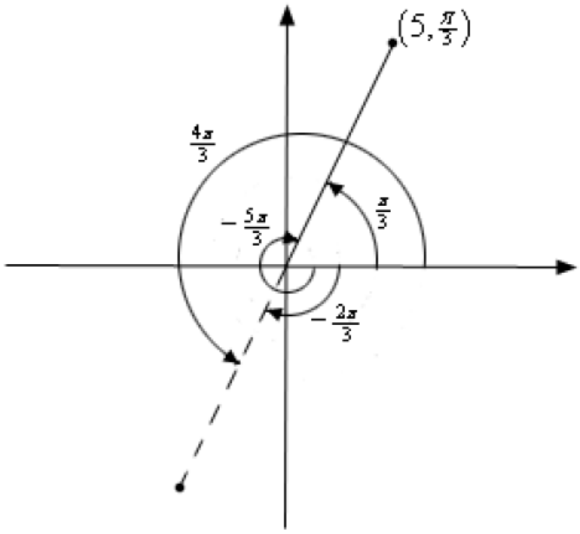
The above discussion may lead one to think that r must be a positive number. However, we also allow r to be negative. Below is a sketch of the two points $\left(2,\frac{\pi}{6}\right)$ and $\left(-2,\frac{\pi}{6}\right)$.



This leads to an important difference between Cartesian coordinates and polar coordinates. In Cartesian coordinates there is exactly one set of coordinates for any given point. With polar coordinates this isn't true. In polar coordinates there is literally an infinite number of coordinates for a given point. For instance, the following four points are all coordinates for the same point.

$$\left(5, \frac{\pi}{3}\right) = \left(5, -\frac{5\pi}{3}\right) = \left(-5, \frac{4\pi}{3}\right) = \left(-5, -\frac{2\pi}{3}\right)$$

Here is a sketch of the angles used in these four sets of coordinates.



These four points only represent the coordinates of the point without rotating around the system more than once. If we allow the angle to make as many complete rotations about the axis system as we want then there are an infinite number of coordinates for the same point. In fact the point (r, θ) can be represented by any of the following coordinate pairs.

$$(r, \theta + 2\pi n) \qquad (-r, \theta + (2n+1)\pi), \qquad \text{where } n \text{ is any integer.}$$

Polar to Cartesian Conversion Formulas

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Converting from Cartesian is almost as easy. Let's first notice the following.

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2\left(\cos^2\theta + \sin^2\theta\right) = r^2 \end{aligned}$$

This is a very useful formula that we should remember, however we are after an equation for r so let's take the square root of both sides. This gives,

$$r = \sqrt{x^2 + y^2}$$

Note that technically we should have a plus or minus in front of the root since we know that r can be either positive or negative. We will run with the convention of positive r here.

Getting an equation for θ is almost as simple. We'll start with,

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

Taking the inverse tangent of both sides gives,

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

We will need to be careful with this because inverse tangents only return values in the range $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Recall that there is a second possible angle and that the second angle is given by $\theta + \pi$.

Summarizing then gives the following formulas for converting from Cartesian coordinates to polar coordinates.

Cartesian to Polar Conversion Formulas

$$r^2 = x^2 + y^2 \qquad\qquad r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example 1 Convert each of the following points into the given coordinate system.

(a) $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates. [Solution]

(b) (-1,-1) into polar coordinates. [Solution]

Solution

(a) Convert $\left(-4, \frac{2\pi}{3}\right)$ into Cartesian coordinates.

This conversion is easy enough. All we need to do is plug the points into the formulas.

$$x = -4\cos\left(\frac{2\pi}{3}\right) = -4\left(-\frac{1}{2}\right) = 2$$

$$y = -4\sin\left(\frac{2\pi}{3}\right) = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

So, in Cartesian coordinates this point is $\left(2, -2\sqrt{3}\right)$.

(b) Convert (-1,-1) into polar coordinates.

Let's first get r.

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

Now, let's get θ.

$$\theta = \tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

This is not the correct angle however. This value of θ is in the first quadrant and the point we've been given is in the third quadrant. As noted above we can get the correct angle by adding π onto this. Therefore, the actual angle is,

$$\theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

So, in polar coordinates the point is $\left(\sqrt{2}, \frac{5\pi}{4}\right)$. Note as well that we could have used the first θ that we got by using a negative r. In this case the point could also be written in polar coordinates as $\left(-\sqrt{2}, \frac{\pi}{4}\right)$.

Example 2 Convert each of the following into an equation in the given coordinate system.

(a) Convert $2x - 5x^3 = 1 + xy$ into polar coordinates. [Solution]

(b) Convert $r = -8\cos\theta$ into Cartesian coordinates. [Solution]

Solution

(a) Convert $2x - 5x^3 = 1 + xy$ into polar coordinates.

In this case there really isn't much to do other than plugging in the formulas for x and y (*i.e.* the Cartesian coordinates) in terms of r and θ (*i.e.* the polar coordinates).

$$2(r\cos\theta) - 5(r\cos\theta)^3 = 1 + (r\cos\theta)(r\sin\theta)$$
$$2r\cos\theta - 5r^3\cos^3\theta = 1 + r^2\cos\theta\sin\theta$$

(b) Convert $r = -8\cos\theta$ into Cartesian coordinates.

This one is a little trickier, but not by much. First notice that we could substitute straight for the r. However, there is no straight substitution for the cosine that will give us only Cartesian coordinates. If we had an r on the right along with the cosine then we could do a direct substitution. So, if an r on the right side would be convenient let's put one there, just don't forget to put one on the left side as well.

$$r^2 = -8r\cos\theta$$

We can now make some substitutions that will convert this into Cartesian coordinates.

$$x^2 + y^2 = -8x$$

2- <u>*Common Polar Coordinate Graphs.*</u>

Let's identify a few of the more common graphs in polar coordinates. We'll also take a look at a couple of special polar graphs.

Lines

Some lines have fairly simple equations in polar coordinates.

1. $\theta = \beta$.

We can see that this is a line by converting to Cartesian coordinates as follows

$$\theta = \beta$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \beta$$

$$\frac{y}{x} = \tan\beta$$

$$y = (\tan\beta)x$$

This is a line that goes through the origin and makes an angle of β with the positive x-axis. Or, in other words it is a line through the origin with slope of $\tan\beta$.

2. $r\cos\theta = a$

This is easy enough to convert to Cartesian coordinates to $x = a$. So, this is a vertical line.

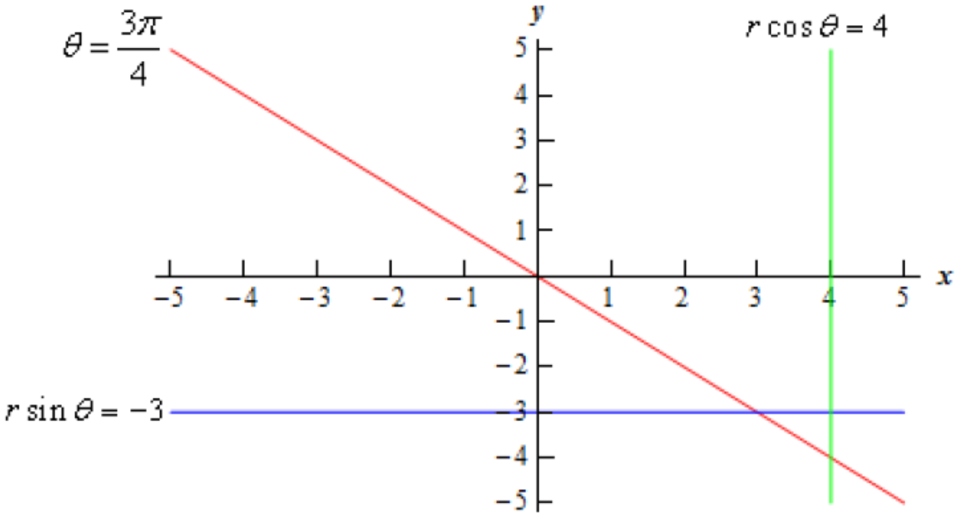
3. $r\sin\theta = b$

Likewise, this converts to $y = b$ and so is a horizontal line.

Example 3 Graph $\theta = \frac{3\pi}{4}$, $r\cos\theta = 4$ and $r\sin\theta = -3$ on the same axis system.

Solution

There really isn't too much to this one other than doing the graph so here it is.



Circles

Let's take a look at the equations of circles in polar coordinates.

1. $r = a$.

   This equation is saying that no matter what angle we've got the distance from the origin must be a. If you think about it that is exactly the definition of a circle of radius a centered at the origin.

   So, this is a circle of radius a centered at the origin. This is also one of the reasons why we might want to work in polar coordinates. The equation of a circle centered at the origin has a very nice equation, unlike the corresponding equation in Cartesian coordinates.

2. $r = 2a\cos\theta$.

   We looked at a specific example of one of these when we were converting equations to Cartesian coordinates.

   This is a circle of radius $|a|$ and center $(a, 0)$. Note that a might be negative (as it was in our example above) and so the absolute value bars are required on the radius. They should not be used however on the center.

3. $r = 2b\sin\theta$.

   This is similar to the previous one. It is a circle of radius $|b|$ and center $(0, b)$.

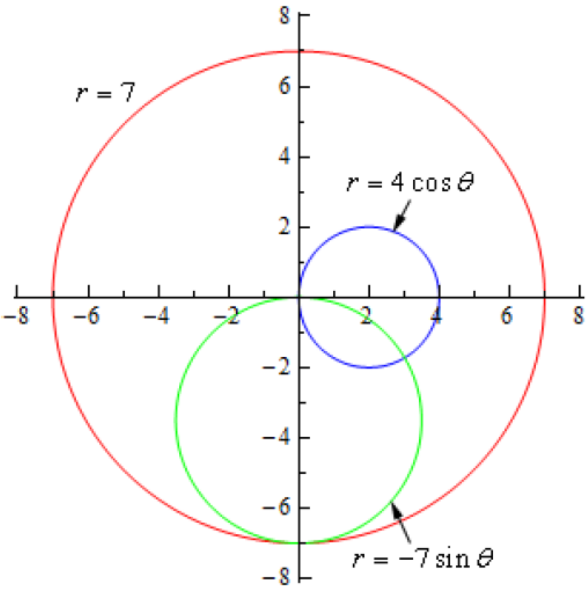
4. $r = 2a\cos\theta + 2b\sin\theta$.

   This is a combination of the previous two and by completing the square twice it can be shown that this is a circle of radius $\sqrt{a^2 + b^2}$ and center (a, b). In other words, this is the general equation of a circle that isn't centered at the origin.

Example 4 Graph $r = 7$, $r = 4\cos\theta$, and $r = -7\sin\theta$ on the same axis system.

Solution

The first one is a circle of radius 7 centered at the origin. The second is a circle of radius 2 centered at (2,0). The third is a circle of radius $\frac{7}{2}$ centered at $\left(0, -\frac{7}{2}\right)$. Here is the graph of the three equations.



Cardioids and Limacons

These can be broken up into the following three cases.

1. Cardioids : $r = a \pm a\cos\theta$ and $r = a \pm a\sin\theta$.
   These have a graph that is vaguely heart shaped and always contain the origin.

2. Limacons with an inner loop : $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ with $a < b$.
   These will have an inner loop and will always contain the origin.

3. Limacons without an inner loop : $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ with $a > b$.
   These do not have an inner loop and do not contain the origin.

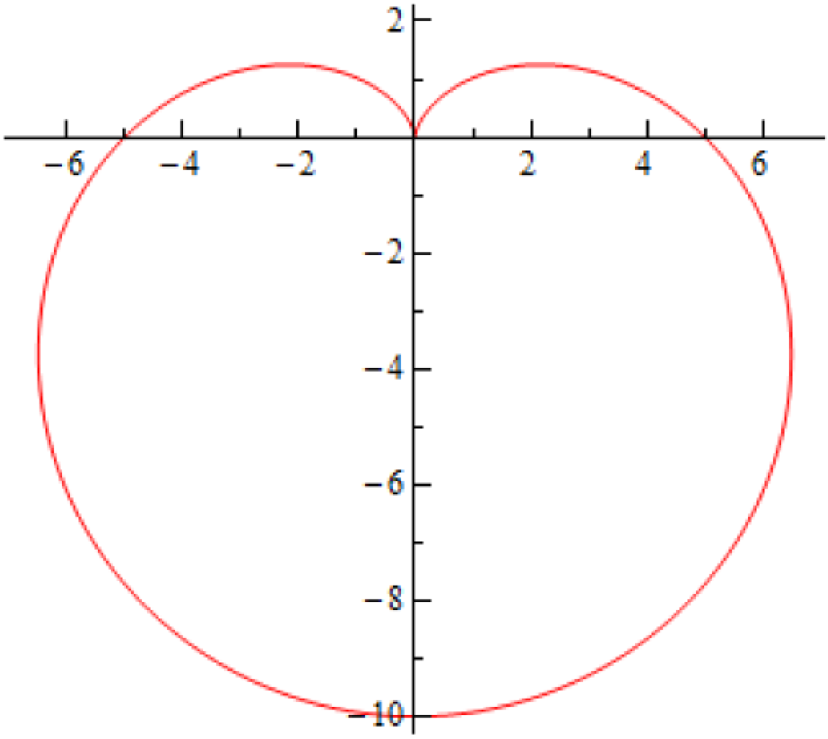
Example 5 Graph $r = 5 - 5\sin\theta$, $r = 7 - 6\cos\theta$, and $r = 2 + 4\cos\theta$.

Solution

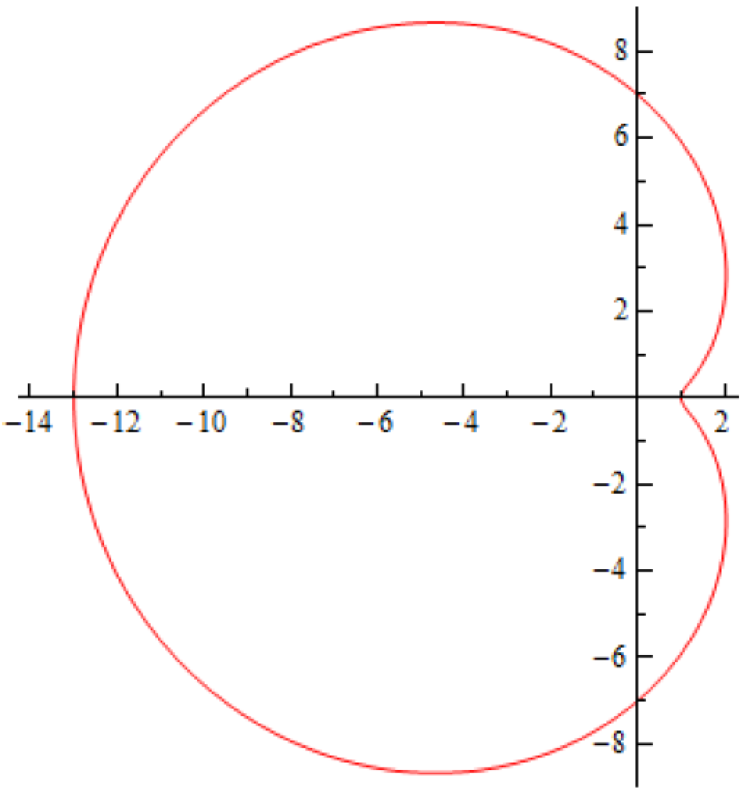
These will all graph out once in the range $0 \le \theta \le 2\pi$. Here is a table of values for each followed by graphs of each.

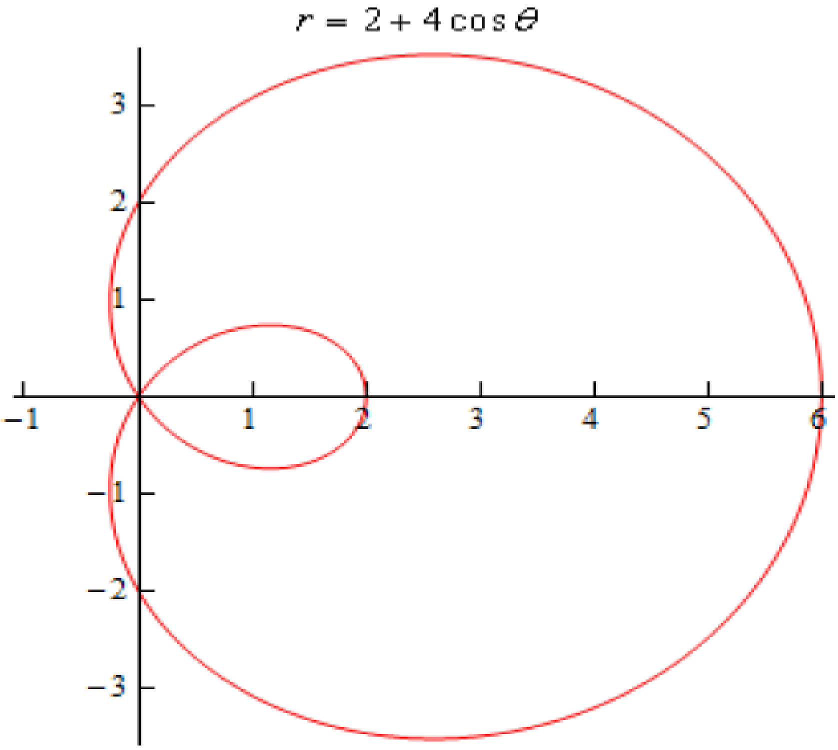
θ	$r = 5 - 5\sin\theta$	$r = 7 - 6\cos\theta$	$r = 2 + 4\cos\theta$
0	5	1	6
$\frac{\pi}{2}$	0	7	2
π	5	13	-2
$\frac{3\pi}{2}$	10	7	2
2π	5	1	6

$r = 5 - 5\sin\theta$



$r = 7 - 6\cos\theta$





3- *Tangents with Polar Coordinates.*

We now need to discuss some calculus topics in terms of polar coordinates.

We will start with finding tangent lines to polar curves. In this case we are going to assume that the equation is in the form $r = f(\theta)$. With the equation in this form we can actually use the equation for the derivative $\frac{dy}{dx}$ we derived when we looked at tangent lines with parametric equations. To do this however requires us to come up with a set of parametric equations to represent the curve. This is actually pretty easy to do.

From our work in the previous section we have the following set of conversion equations for going from polar coordinates to Cartesian coordinates.

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$

Now, we'll use the fact that we're assuming that the equation is in the form $r = f(\theta)$. Substituting this into these equations gives the following set of parametric equations (with θ as the parameter) for the curve.

$$x = f(\theta)\cos\theta \qquad\qquad y = f(\theta)\sin\theta$$

Now, we will need the following derivatives.

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \qquad\qquad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$

$$= \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad\qquad = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

The derivative $\frac{dy}{dx}$ is then,

Derivative with Polar Coordinates

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Example 1 Determine the equation of the tangent line to $r = 3 + 8\sin\theta$ at $\theta = \frac{\pi}{6}$.

Solution
We'll first need the following derivative.

$$\frac{dr}{d\theta} = 8\cos\theta$$

The formula for the derivative $\frac{dy}{dx}$ becomes,

$$\frac{dy}{dx} = \frac{8\cos\theta\sin\theta + (3 + 8\sin\theta)\cos\theta}{8\cos^2\theta - (3 + 8\sin\theta)\sin\theta} = \frac{16\cos\theta\sin\theta + 3\cos\theta}{8\cos^2\theta - 3\sin\theta - 8\sin^2\theta}$$

The slope of the tangent line is,

$$m = \left.\frac{dy}{dx}\right|_{\theta = \frac{\pi}{6}} = \frac{4\sqrt{3} + \frac{3\sqrt{3}}{2}}{4 - \frac{3}{2}} = \frac{11\sqrt{3}}{5}$$

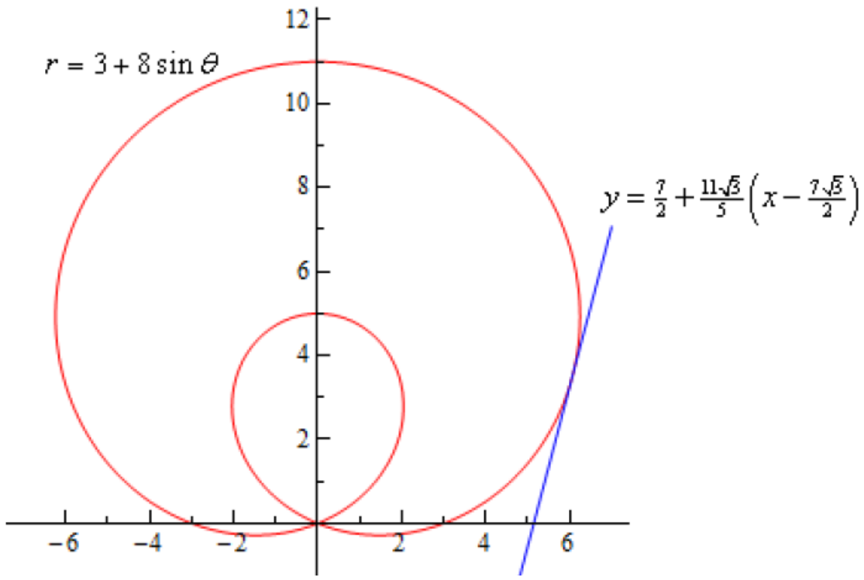
Now, at $\theta = \frac{\pi}{6}$ we have $r = 7$. We'll need to get the corresponding x-y coordinates so we can get the tangent line.

$$x = 7\cos\left(\frac{\pi}{6}\right) = \frac{7\sqrt{3}}{2} \qquad\qquad y = 7\sin\left(\frac{\pi}{6}\right) = \frac{7}{2}$$

The tangent line is then,

$$y = \frac{7}{2} + \frac{11\sqrt{3}}{5}\left(x - \frac{7\sqrt{3}}{2}\right)$$

For the sake of completeness here is a graph of the curve and the tangent line.



4- ***Arc Length with Polar Coordinates.***

In this section we'll look at the arc length of the curve given by,

$$r = f(\theta) \qquad\qquad \alpha \le \theta \le \beta$$

where we also assume that the curve is traced out exactly once. Just as we did with the tangent lines in polar coordinates we'll first write the curve in terms of a set of parametric equations,

$$\begin{aligned} x &= r\cos\theta & \qquad y &= r\sin\theta \\ &= f(\theta)\cos\theta & &= f(\theta)\sin\theta \end{aligned}$$

and we can now use the parametric formula for finding the arc length.

We'll need the following derivatives for these computations.

$$\begin{aligned} \frac{dx}{d\theta} &= f'(\theta)\cos\theta - f(\theta)\sin\theta & \qquad \frac{dy}{d\theta} &= f'(\theta)\sin\theta + f(\theta)\cos\theta \\ &= \frac{dr}{d\theta}\cos\theta - r\sin\theta & &= \frac{dr}{d\theta}\sin\theta + r\cos\theta \end{aligned}$$

We'll need the following for our ds.

$$\begin{aligned} \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2 \\ &= \left(\frac{dr}{d\theta}\right)^2\cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta \\ &\qquad + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2\left(\cos^2\theta + \sin^2\theta\right) + r^2\left(\cos^2\theta + \sin^2\theta\right) \\ &= r^2 + \left(\frac{dr}{d\theta}\right)^2 \end{aligned}$$

The arc length formula for polar coordinates is then,

$$L = \int ds$$

where,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 1 Determine the length of $r = \theta \quad 0 \le \theta \le 1$.

Solution

Okay, let's just jump straight into the formula since this is a fairly simple function.

$$L = \int_0^1 \sqrt{\theta^2 + 1}\, d\theta$$

We'll need to use a trig substitution here.

$$\theta = \tan x \qquad\qquad d\theta = \sec^2 x\, dx$$

$$\theta = 0 \qquad 0 = \tan x \qquad x = 0$$

$$\theta = 1 \qquad 1 = \tan x \qquad x = \frac{\pi}{4}$$

$$\sqrt{\theta^2 + 1} = \sqrt{\tan^2 x + 1} = \sqrt{\sec^2 x} = \left|\sec x\right| = \sec x$$

The arc length is then,

$$\begin{aligned} L &= \int_0^1 \sqrt{\theta^2 + 1}\, d\theta \\ &= \int_0^{\frac{\pi}{4}} \sec^3 x\, dx \\ &= \frac{1}{2}\left(\sec x \tan x + \ln\left|\sec x + \tan x\right|\right)\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right) \end{aligned}$$

5- ***Area Polar Coordinates.***

The equation of a curve in polar coordinates is given by $r = f(\theta)$. To find the area bounded by the curve $r = f(\theta)$, the rays $\theta = \alpha$ and $\theta = \beta$, divide the angle $\beta - \alpha$ into n-parts by defining $\Delta\theta = \frac{\beta-\alpha}{n}$ and then defining the rays

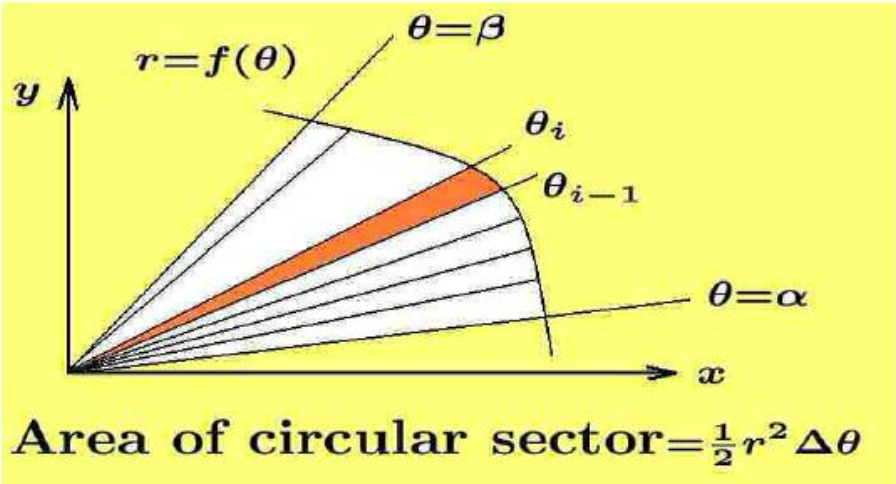
$$\theta_0 = \alpha,\ \theta_1 = \theta_0 + \Delta\theta, \ldots, \theta_i = \theta_{i-1} + \Delta\theta, \ldots, \theta_n = \theta_{n-1} + \Delta\theta = \beta$$

The area between the rays $\theta = \theta_{i-1}$, $\theta = \theta_i$ and the curve $r = f(\theta)$, illustrated in the figure below, is approximated by a circular sector with area element

$$dA_i = \frac{1}{2} r_i^2 \Delta\theta_i = \frac{1}{2} f^2(\theta_i)\,\Delta\theta_i$$

where $\Delta\theta_i = \theta_i - \theta_{i-1}$ and $r_i = f(\theta_i)$. A summation of these elements of area between the rays $\theta = \alpha$ and $\theta = \beta$ gives the approximate area

$$\sum_{i=1}^{n} dA_i = \sum_{i=1}^{n} \frac{1}{2} r_i^2 \Delta\theta_i = \sum_{i=1}^{n} \frac{1}{2} f^2(\theta_i)\,\Delta\theta_i$$



Approximation of area by summation of circular sectors.

This approximation gets better as $\Delta\theta_i$ gets smaller. Using the fundamental theorem of integral calculus, it can be shown that in the limit as $n \to \infty$, the equation defines the element of area $dA = \frac{1}{2} r^2\, d\theta$. A summation of these elements of area gives

$$\text{Polar Area} = \int_\alpha^\beta dA = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta = \frac{1}{2}\int_\alpha^\beta f^2(\theta)\, d\theta$$

Example 1. Find the area bounded by the polar curve

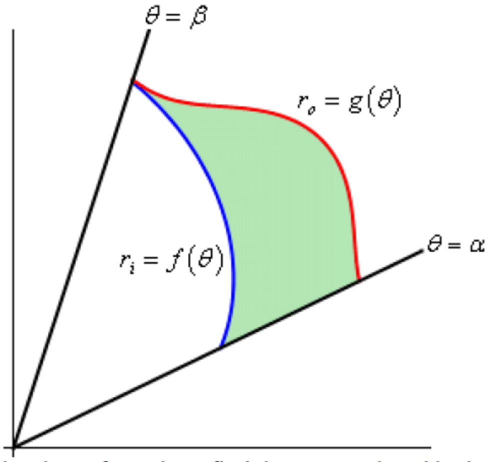
$$r = 2r_0 \cos\theta \quad \text{for } 0 \le \theta \le \pi.$$

Solution

One finds that the polar curve $r = 2r_0 \cos\theta$, for $0 \le \theta \le \pi$, is a circle of radius r_0 which has its center at the point $(r_0, 0)$ in polar coordinates. Using the area formula given by equation (3.121) one obtains

$$\text{Area} = \frac{1}{2}\int_0^{\pi} (2r_0\cos\theta)^2\,d\theta = 2r_0^2 \int_0^{\pi} \cos^2\theta\,d\theta = r_0^2 \int_0^{\pi} (\cos 2\theta + 1)\,d\theta = r_0^2 \left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\pi} = \pi r_0^2$$

So, that's how we determine areas that are enclosed by a single curve, but what about situations like the following sketch were we want to find the area between two curves.



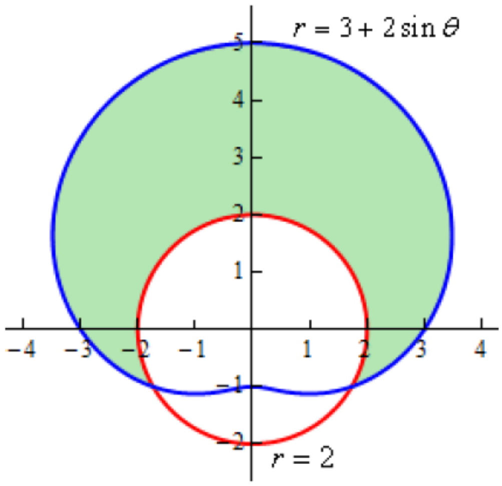
In this case we can use the above formula to find the area enclosed by both and then the actual area is the difference between the two. The formula for this is,

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_o^2 - r_i^2\right) d\theta}$$

Example 2 Determine the area that lies inside $r = 3 + 2\sin\theta$ and outside $r = 2$.

Solution

Here is a sketch of the region that we are after.

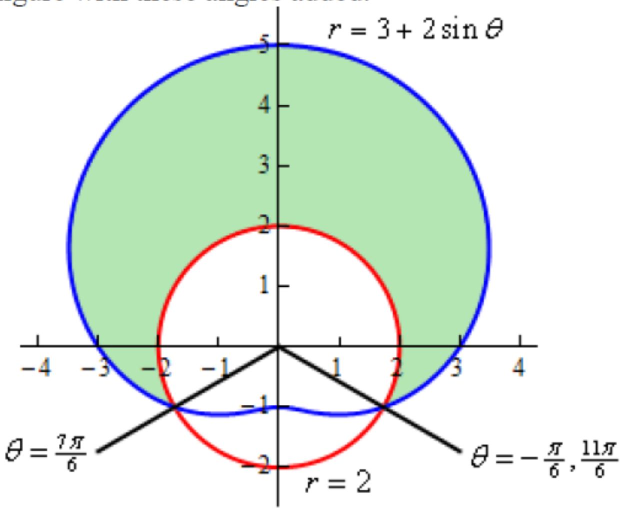


To determine this area we'll need to know the values of θ for which the two curves intersect. We can determine these points by setting the two equations and solving.

$$3 + 2\sin\theta = 2$$

$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



Note as well here that we also acknowledged that another representation for the angle $\frac{11\pi}{6}$ is $-\frac{\pi}{6}$. This is important for this problem. In order to use the formula above the area must be enclosed as we increase from the smaller to larger angle. So, if we use $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we will not enclose the shaded area, instead we will enclose the bottom most of the three regions. However if we use the angles $-\frac{\pi}{6}$ to $\frac{7\pi}{6}$ we will enclose the area that we're after.

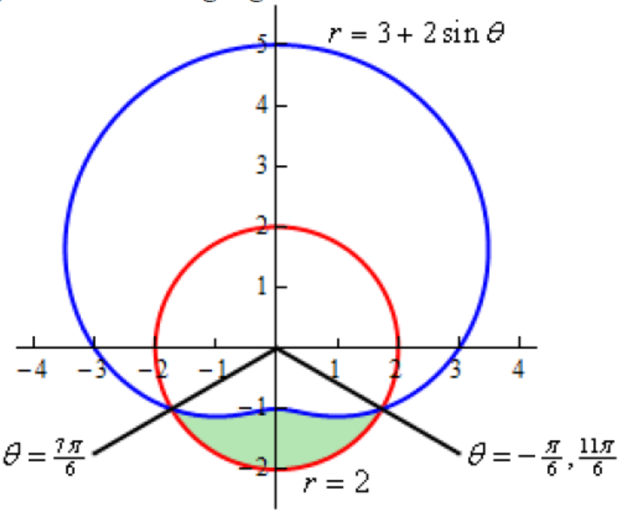
So, the area is then,

$$\begin{aligned} A &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left((3+2\sin\theta)^2 - (2)^2\right)d\theta \\ &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(5+12\sin\theta+4\sin^2\theta\right)d\theta \\ &= \int_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}\left(7+12\sin\theta-2\cos(2\theta)\right)d\theta \\ &= \frac{1}{2}\left(7\theta-12\cos\theta-\sin(2\theta)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\ &= \frac{11\sqrt{3}}{2}+\frac{14\pi}{3} = 24.187 \end{aligned}$$

Example 3 Determine the area of the region outside $r = 3 + 2\sin\theta$ and inside $r = 2$.

Solution

This time we're looking for the following region.



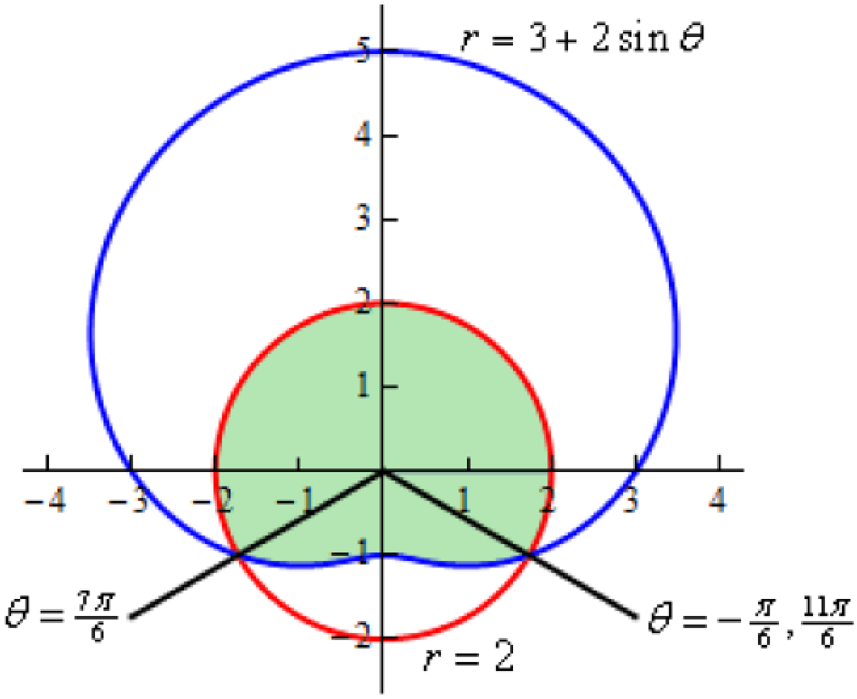
So, this is the region that we get by using the limits $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$. The area for this region is,

$$\begin{aligned}
A &= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left((2)^2 - (3 + 2\sin\theta)^2\right) d\theta \\
&= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-5 - 12\sin\theta - 4\sin^2\theta\right) d\theta \\
&= \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \frac{1}{2}\left(-7 - 12\sin\theta + 2\cos(2\theta)\right) d\theta \\
&= \frac{1}{2}\left(-7\theta + 12\cos\theta + \sin(2\theta)\right)\Big|_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \\
&= \frac{11\sqrt{3}}{2} - \frac{7\pi}{3} = 2.196
\end{aligned}$$

Example 4 Determine the area that is inside both $r = 3 + 2\sin\theta$ and $r = 2$.

Solution

Here is the sketch for this example.



In this case however, that is not a major problem. There are two ways to do get the area in this problem. We'll take a look at both of them.

Solution 1

In this case let's notice that the circle is divided up into two portions and we're after the upper portion. Also notice that we found the area of the lower portion in Example 3. Therefore, the area is,

$$\begin{aligned} \text{Area} &= \text{Area of Circle} - \text{Area from Example 3} \\ &= \pi(2)^2 - 2.196 \\ &= 10.370 \end{aligned}$$

$$\begin{aligned} \text{Area} &= \text{Area of Limacon} - \text{Area from Example 2} \\ &= \int_0^{2\pi} \frac{1}{2}(3 + 2\sin\theta)^2\, d\theta - 24.187 \\ &= \int_0^{2\pi} \frac{1}{2}\left(9 + 12\sin\theta + 4\sin^2\theta\right) d\theta - 24.187 \\ &= \int_0^{2\pi} \frac{1}{2}\left(11 + 12\sin\theta - 2\cos(2\theta)\right) d\theta - 24.187 \\ &= \frac{1}{2}\left(11\theta - 12\cos(\theta) - \sin(2\theta)\right)\Big|_0^{2\pi} - 24.187 \\ &= 11\pi - 24.187 \\ &= 10.370 \end{aligned}$$

6- ***Problems.***

A- ***Polar Coordinates.***

1. For the point with polar coordinates $\left(2, \frac{\pi}{7}\right)$ determine three different sets of coordinates for the same point all of which have angles different from $\frac{\pi}{5}$ and are in the range $-2\pi \le \theta \le 2\pi$.

2. The polar coordinates of a point are $(-5, 0.23)$. Determine the Cartesian coordinates for the point.

3. The Cartesian coordinate of a point are $(2, -6)$. Determine a set of polar coordinates for the point.

4. The Cartesian coordinate of a point are $(-8, 1)$. Determine a set of polar coordinates for the point.

For problems 5 and 6 convert the given equation into an equation in terms of polar coordinates.

5. $\dfrac{4x}{3x^2 + 3y^2} = 6 - xy$

6. $x^2 = \dfrac{4x}{y} - 3y^2 + 2$

For problems 7 and 8 convert the given equation into an equation in terms of Cartesian coordinates.

7. $6r^3 \sin\theta = 4 - \cos\theta$

8. $\dfrac{2}{r} = \sin\theta - \sec\theta$

For problems 9 – 16 sketch the graph of the given polar equation.

9. $\cos\theta = \dfrac{6}{r}$

10. $\theta = -\dfrac{\pi}{3}$

11. $r = -14\cos\theta$

12. $r = 7$

13. $r = 9\sin\theta$

14. $r = 8 + 8\cos\theta$

15. $r = 5 - 2\sin\theta$

16. $r = 4 - 9\sin\theta$

B- ***Tangents with Polar Coordinates*.**

1. Find the tangent line to $r = \sin(4\theta)\cos(\theta)$ at $\theta = \dfrac{\pi}{6}$.

2. Find the tangent line to $r = \theta - \cos(\theta)$ at $\theta = \dfrac{3\pi}{4}$.

C- ***Area with Polar Coordinates*.**

1. Find the area inside the inner loop of $r = 3 - 8\cos\theta$.

2. Find the area inside the graph of $r = 7 + 3\cos\theta$ and to the left of the y-axis.

3. Find the area that is inside $r = 3 + 3\sin\theta$ and outside $r = 2$.

4. Find the area that is inside $r = 2$ and outside $r = 3 + 3\sin\theta$.

5. Find the area that is inside $r = 4 - 2\cos\theta$ and outside $r = 6 + 2\cos\theta$.

6. Find the area that is inside both $r = 1 - \sin\theta$ and $r = 2 + \sin\theta$.

D- ***Arc Length with Polar Coordinates*.**

1. Determine the length of the following polar curve. You may assume that the curve traces out exactly once for the given range of θ.

$$r = -4\sin\theta\,,\ 0 \le \theta \le \pi$$

For problems 2 and 3 set up, but do not evaluate, an integral that gives the length of the given polar curve. For these problems you may assume that the curve traces out exactly once for the given range of θ.

2. $r = \theta\cos\theta\,,\ 0 \le \theta \le \pi$

3. $r = \cos(2\theta) + \sin(3\theta)\,,\ 0 \le \theta \le 2\pi$

By

Ms.C. Yasir R. Al-hamdany

Lecture – Five

***Sequences* and Series**

Sequences and Series →

- *Sequences.*
- *Terminology and Definitions.*
- *Series – The Basics.*
- *Type of Series.*
- *Comparison Test.*
- *Absolute Convergence.*
- *Ratio Test.*
- *Root Test.*
- *Taylor Series.*
- *Problems.*

1- Sequences.

A sequence is nothing more than a list of numbers written in a specific order. General sequence terms are denoted as follows,

$$\begin{aligned} &a_1 - \text{first term} \\ &a_2 - \text{second term} \\ &\quad\vdots \\ &a_n - n^{th} \text{ term} \\ &a_{n+1} - (n+1)^{\text{st}} \text{ term} \end{aligned}$$

There is a variety of ways of denoting a sequence. Each of the following are equivalent ways of denoting a sequence.

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

Example 1 Write down the first few terms of each of the following sequences.

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$ [Solution]

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$ [Solution]

(c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π [Solution]

(a) $\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$

To get the first few sequence terms here all we need to do is plug in values of n into the formula given and we'll get the sequence terms.

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty} = \left\{\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \ldots\right\}$$

(b) $\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$

This one is similar to the first one. The main difference is that this sequence doesn't start at $n = 1$.

$$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty} = \left\{-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots\right\}$$

(c) $\{b_n\}_{n=1}^{\infty}$, where $b_n = n^{th}$ digit of π

So we know that $\pi = 3.14159265359\ldots$

The sequence is then,

$$\{3,1,4,1,5,9,2,6,5,3,5,\ldots\}$$

Theorem 1

Given the sequence $\{a_n\}$ if we have a function $f(x)$ such that $f(n) = a_n$ and $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{n\to\infty} a_n = L$

Theorem 2

If $\lim\limits_{n\to\infty} |a_n| = 0$ then $\lim\limits_{n\to\infty} a_n = 0$.

Theorem 3

The sequence $\{r^n\}_{n=0}^{\infty}$ converges if $-1 < r \le 1$ and diverges for all other value of r. Also,

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Example 2 Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

(a) $\left\{\dfrac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$ [Solution]

(b) $\left\{\dfrac{\mathbf{e}^{2n}}{n}\right\}_{n=1}^{\infty}$ [Solution]

(c) $\left\{\dfrac{(-1)^n}{n}\right\}_{n=1}^{\infty}$ [Solution]

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$ [Solution]

Solution

(a) $\left\{\dfrac{3n^2-1}{10n+5n^2}\right\}_{n=2}^{\infty}$

To do a limit in this form all we need to do is factor from the numerator and denominator the largest power of n, cancel and then take the limit.

$$\lim_{n\to\infty}\frac{3n^2-1}{10n+5n^2}=\lim_{n\to\infty}\frac{n^2\left(3-\frac{1}{n^2}\right)}{n^2\left(\frac{10}{n}+5\right)}=\lim_{n\to\infty}\frac{3-\frac{1}{n^2}}{\frac{10}{n}+5}=\frac{3}{5}$$

So the sequence converges and its limit is $\frac{3}{5}$.

(b) $\left\{\frac{\mathbf{e}^{2n}}{n}\right\}_{n=1}^{\infty}$

Normally this would be a problem, but we've got Theorem 1 from above to help us out. Let's define

$$f(x)=\frac{\mathbf{e}^{2x}}{x}$$

and note that,

$$f(n)=\frac{\mathbf{e}^{2n}}{n}$$

Theorem 1 says that all we need to do is take the limit of the function.

$$\lim_{n\to\infty}\frac{\mathbf{e}^{2n}}{n}=\lim_{x\to\infty}\frac{\mathbf{e}^{2x}}{x}=\lim_{x\to\infty}\frac{2\mathbf{e}^{2x}}{1}=\infty$$

So, the sequence in this part diverges (to ∞).

(c) $\left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty}$

We will need to use Theorem 2 on this problem. To this

$$\lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right|=\lim_{n\to\infty}\frac{1}{n}=0$$

Therefore, since the limit of the sequence terms with absolute value bars on them goes to zero we know by Theorem 2 that,

$$\lim_{n\to\infty}\frac{(-1)^n}{n}=0$$

which also means that the sequence converges to a value of zero.

(d) $\left\{(-1)^n\right\}_{n=0}^{\infty}$

For this theorem note that all we need to do is realize that this is the sequence in Theorem 3 above using $r=-1$. So, by Theorem 3 this sequence diverges.

Theorem 4

For the sequence $\{a_n\}$ if both $\lim_{n\to\infty} a_{2n}=L$ and $\lim_{n\to\infty} a_{2n+1}=L$ then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n=L$.

2- *Terminology and Definitions.*

Let's start off with some terminology and definitions.

Given any sequence $\{a_n\}$ we have the following.

1. We call the sequence **increasing** if $a_n < a_{n+1}$ for every n.
2. We call the sequence **decreasing** if $a_n > a_{n+1}$ for every n.
3. If $\{a_n\}$ is an increasing sequence or $\{a_n\}$ is a decreasing sequence we call it **monotonic**.
4. If there exists a number m such that $m \le a_n$ for every n we say the sequence is **bounded below**. The number m is sometimes called a **lower bound** for the sequence.
5. If there exists a number M such that $a_n \le M$ for every n we say the sequence is **bounded above**. The number M is sometimes called an **upper bound** for the sequence.
6. If the sequence is both bounded below and bounded above we call the sequence **bounded**.

Example 1 Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$ [Solution]

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$ [Solution]

(c) $\left\{\frac{2}{n^2}\right\}_{n=5}^{\infty}$ [Solution]

Solution

(a) $\left\{-n^2\right\}_{n=0}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) because,

$$-n^2 > -(n+1)^2$$

for every n.

(b) $\left\{(-1)^{n+1}\right\}_{n=1}^{\infty}$

The sequence terms in this sequence alternate between 1 and -1 and so the sequence is neither an increasing sequence or a decreasing sequence. Since the sequence is neither an increasing nor decreasing sequence it is not a monotonic sequence.

The sequence is bounded however since it is bounded above by 1 and bounded below by -1.

(c) $\left\{\frac{2}{n^2}\right\}_{n=5}^{\infty}$

This sequence is a decreasing sequence (and hence monotonic) since,

$$\frac{2}{n^2} > \frac{2}{(n+1)^2}$$

The terms in this sequence are all positive and so it is bounded below by zero. Also, since the sequence is a decreasing sequence the first sequence term will be the largest and so we can see that the sequence will also be bounded above by $\frac{2}{25}$. Therefore, this sequence is bounded.

We can also take a quick limit and note that this sequence converges and its limit is zero.

Example 2 Determine if the following sequences are monotonic and/or bounded.

(a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ [Solution]

(b) $\left\{\frac{n^3}{n^4+10000}\right\}_{n=0}^{\infty}$ [Solution]

Solution

(a) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

To determine the increasing/decreasing nature of this sequence we will need to resort to Calculus I techniques. First consider the following function and its derivative.

$$f\left(x\right)=\frac{x}{x+1} \qquad\qquad f'\left(x\right)=\frac{1}{\left(x+1\right)^2}$$

We can see that the first derivative is always positive and so from Calculus I we know that the function must then be an increasing function. So, how does this help us? Notice that,

$$f\left(n\right)=\frac{n}{n+1}=a_n$$

Therefore because $n<n+1$ and $f\left(x\right)$ is increasing we can also say that,

$$a_n=\frac{n}{n+1}=f\left(n\right)<f\left(n+1\right)=\frac{n+1}{n+1+1}=a_{n+1} \qquad \Rightarrow \qquad a_n<a_{n+1}$$

In other words, the sequence must be increasing.

Note that now that we know the sequence is an increasing sequence we can get a better lower bound for the sequence. Since the sequence is increasing the first term in the sequence must be the smallest term and so since we are starting at $n=1$ we could also use a lower bound of $\frac{1}{2}$ for this sequence. It is important to remember that any number that is always less than or equal to all the sequence terms can be a lower bound. Some are better than others however.

A quick limit will also tell us that this sequence converges with a limit of 1.

(b) $\left\{\dfrac{n^3}{n^4+10000}\right\}_{n=0}^{\infty}$

This however, isn't a decreasing sequence. Let's take a look at the first few terms to see this.

$$a_1=\frac{1}{10001}\approx 0.00009999 \qquad a_2=\frac{1}{1252}\approx 0.0007987$$

$$a_3=\frac{27}{10081}\approx 0.005678 \qquad a_4=\frac{4}{641}\approx 0.006240$$

$$a_5=\frac{1}{85}\approx 0.011756 \qquad a_6=\frac{27}{1412}\approx 0.019122$$

$$a_7=\frac{343}{12401}\approx 0.02766 \qquad a_8=\frac{32}{881}\approx 0.03632$$

$$a_9=\frac{729}{16561}\approx 0.04402 \qquad a_{10}=\frac{1}{20}=0.05$$

Now, we can't make another common mistake and assume that because the first few terms increase then whole sequence must also increase. If we did that we would also be mistaken as this is also not an increasing sequence.

This sequence is neither decreasing or increasing. The only sure way to see this is to do the Calculus I approach to increasing/decreasing functions.

In this case we'll need the following function and its derivative.

$$f(x) = \frac{x^3}{x^4 + 10000} \qquad\qquad f'(x) = \frac{-x^2\left(x^4 - 30000\right)}{\left(x^4 + 10000\right)^2}$$

This function will have the following three critical points,

$$x = 0, \;\; x = \sqrt[4]{30000} \approx 13.1607, \quad x = -\sqrt[4]{30000} \approx -13.1607$$

Why critical points? Remember these are the only places where the function *may* change sign! Our sequence starts at $n = 0$ and so we can ignore the third one since it lies outside the values of n that we're considering. By plugging in some test values of x we can quickly determine that the derivative is positive for $0 < x < \sqrt[4]{30000} \approx 13.16$ and so the function is increasing in this range. Likewise, we can see that the derivative is negative for $x > \sqrt[4]{30000} \approx 13.16$ and so the function will be decreasing in this range.

So, our sequence will be increasing for $0 \le n \le 13$ and decreasing for $n \ge 13$. Therefore the function is not monotonic.

3- Series – The Basics.

That topic is infinite series. So just what is an infinite series? Well, let's start with a sequence $\left\{a_n\right\}_{n=1}^{\infty}$ (note the $n = 1$ is for convenience, it can be anything) and define the following,

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
s_4 &= a_1 + a_2 + a_3 + a_4 \\
&\;\vdots \\
s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{n} a_i
\end{aligned}$$

The s_n are called **partial sums** and notice that they will form a sequence, $\left\{s_n\right\}_{n=1}^{\infty}$. Also recall that the Σ is used to represent this summation and called a variety of names. The most common names are : **series notation**, **summation notation**, and **sigma notation**.

We want to take a look at the limit of the sequence of partial sums, $\left\{s_n\right\}_{n=1}^{\infty}$.

Notationally we'll define,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$$

If the sequence of partial sums, $\{s_n\}_{n=1}^{\infty}$, is convergent and its limit is finite then we also call the infinite series, $\sum_{i=1}^{\infty} a_i$ **convergent** and if the sequence of partial sums is divergent then the infinite series is also called **divergent**.

Note that sometimes it is convenient to write the infinite series as,

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

So, we've determined the convergence of four series now. Two of the series converged and two diverged. Let's go back and examine the series terms for each of these. For each of the series let's take the limit as n goes to infinity of the series terms (not the partial sums!!).

$\lim_{n\to\infty} n = \infty$	this series diverged
$\lim_{n\to\infty} \frac{1}{n^2-1} = 0$	this series converged
$\lim_{n\to\infty} (-1)^n$ doesn't exist	this series diverged
$\lim_{n\to\infty} \frac{1}{3^{n-1}} = 0$	this series converged

Theorem

If $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$.

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ will diverge.

Example -1 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$$

That's what we'll do here.

$$\lim_{n\to\infty}\frac{4n^2-n^3}{10+2n^3}=-\frac{1}{2}\neq 0$$

The limit of the series terms isn't zero and so by the Divergence Test the series diverges.

4- ***Type of Series.***

A- ***Geometric Series.***

A geometric series is any series that can be written in the form,

$$\sum_{n=1}^{\infty} ar^{n-1}$$

or, with an index shift the geometric series will often be written as,

$$\sum_{n=0}^{\infty} ar^{n}$$

These are identical series and will have identical values, provided they converge of course.

Recall that by multiplying s_n by r and subtracting the result from s_n one obtains

If we start with the first form it can be shown that the partial sums are,

$$s_n=\frac{a\left(1-r^n\right)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}$$

The series will converge provided the partial sums form a convergent sequence, so let's take the limit of the partial sums.

$$\begin{aligned}\lim_{n\to\infty}s_n&=\lim_{n\to\infty}\left(\frac{a}{1-r}-\frac{ar^n}{1-r}\right)\\&=\lim_{n\to\infty}\frac{a}{1-r}-\lim_{n\to\infty}\frac{ar^n}{1-r}\\&=\frac{a}{1-r}-\frac{a}{1-r}\lim_{n\to\infty}r^n\end{aligned}$$

Now, from Theorem 3 from the Sequences section we know that the limit above will exist and be finite provided $-1<r\le 1$. However, note that we can't let $r=1$ since this will give division by zero. Therefore, this will exist and be finite provided $-1<r<1$ and in this case the limit is zero and so we get,

$$\lim_{n\to\infty}s_n=\frac{a}{1-r}$$

Therefore, a geometric series will converge if $-1<r<1$, which is usually written $|r|<1$, its value is,

$$\sum_{n=1}^{\infty}ar^{n-1}=\sum_{n=0}^{\infty}ar^{n}=\frac{a}{1-r}$$

Example 1 Determine if the following series converge or diverge. If they converge give the value of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$

So, let's first get rid of that.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 9^{-(n-2)}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}}$$

Now let's get the correct exponent on each of the numbers. This can be done using simple exponent properties.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} = \sum_{n=1}^{\infty} \frac{4^{n-1}4^{2}}{9^{n-1}9^{-1}}$$

Now, rewrite the term a little.

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 16(9)\frac{4^{n-1}}{9^{n-1}} = \sum_{n=1}^{\infty} 144\left(\frac{4}{9}\right)^{n-1}$$

So, this is a geometric series with $a = 144$ and $r = \frac{4}{9} < 1$. Therefore, since $|r| < 1$ we know the series will converge and its value will be,

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = \frac{144}{1-\frac{4}{9}} = \frac{9}{5}(144) = \frac{1296}{5}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$

$$\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}} = \sum_{n=0}^{\infty} \frac{\left((-4)^3\right)^n}{5^n 5^{-1}} = \sum_{n=0}^{\infty} 5\frac{(-64)^n}{5^n} = \sum_{n=0}^{\infty} 5\left(\frac{-64}{5}\right)^n$$

So, we've got it into the correct form and we can see that $a = 5$ and $r = -\frac{64}{5}$. Also note that $|r| \geq 1$ and so this series diverges.

Example 2 Use the results from the previous example to determine the value of the following series.

(a) $\sum_{n=0}^{\infty} 9^{-n+2} 4^{n+1}$

(b) $\sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1}$

Solution

(a) $\sum_{n=0}^{\infty} 9^{-n+2} 4^{n+1}$

Let's notice that if we strip out the first term from this series we arrive at,

$$\sum_{n=0}^{\infty} 9^{-n+2} 4^{n+1} = 9^2 4^1 + \sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = 324 + \sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1}$$

From the previous example we know the value of the new series that arises here and so the value of the series in this example is,

$$\sum_{n=0}^{\infty} 9^{-n+2} 4^{n+1} = 324 + \frac{1296}{5} = \frac{2916}{5}$$

(b) $\sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1}$

$$\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = 9^1 4^2 + 9^0 4^3 + \sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1} = 208 + \sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1}$$

We can now use the value of the series from the previous example to get the value of this series.

$$\sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1} = \sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} - 208 = \frac{1296}{5} - 208 = \frac{256}{5}$$

B- ***Power Series***.

Fact (The p –series Test)

If $k > 0$ then $\sum_{n=k}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

Using the p-series test makes it very easy to determine the convergence of some series.

Example 3 Determine if the following series are convergent or divergent.

(a) $\sum_{n=4}^{\infty} \frac{1}{n^7}$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Solution

(a) In this case $p = 7 > 1$ and so by this fact the series is convergent.

(b) For this series $p = \frac{1}{2} \le 1$ and so the series is divergent by the fact.

In this section we are going to start talking about power series. A **power series about a**, or just **power series**, is any series that can be written in the form,

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

The c_n's are often called the **coefficients** of the series.

First, as we will see in our examples, we will be able to show that there is a number R so that the power series will converge for, $|x-a| < R$ and will diverge for $|x-a| > R$. This number is called the **radius of convergence** for the series. Note that the series may or may not converge if $|x-a| = R$. What happens at these points will not change the radius of convergence.

Secondly, the interval of all x's, including the endpoints if need be, for which the power series converges is called the **interval of convergence** of the series.

These two concepts are fairly closely tied together. If we know that the radius of convergence of a power series is R then we have the following.

$a - R < x < a + R$	power series converges
$x < a - R$ and $x > a + R$	power series diverges

The interval of convergence must then contain the interval $a - R < x < a + R$ since we know that the power series will converge for these values.

Before getting into some examples let's take a quick look at the convergence of a power series for the case of $x = a$. In this case the power series becomes,

$$\sum_{n=0}^{\infty} c_n (a-a)^n = \sum_{n=0}^{\infty} c_n (0)^n = c_0 (0)^0 + \sum_{n=1}^{\infty} c_n (0)^n = c_0 + \sum_{n=1}^{\infty} 0 = c_0 + 0 = c_0$$

and so the power series converges. Note that we had to strip out the first term since it was the only non-zero term in the series.

Example 1 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} (x+3)^n$$

Solution

With all that said, the best tests to use here are almost always the ratio or root test. Most of the power series that we'll be looking at are set up for one or the other. In this case we'll use the ratio test.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (n+1)(x+3)^{n+1}}{4^{n+1}} \frac{4^n}{(-1)^n (n)(x+3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-(n+1)(x+3)}{4n} \right|$$

The limit is then,

$$L = |x+3| \lim_{n\to\infty} \frac{n+1}{4n}$$

$$= \frac{1}{4}|x+3|$$

So, the ratio test tells us that if $L < 1$ the series will converge, if $L > 1$ the series will diverge, and if $L = 1$ we don't know what will happen. So, we have,

$$\frac{1}{4}|x+3| < 1 \quad \Rightarrow \quad |x+3| < 4 \qquad \text{series converges}$$

$$\frac{1}{4}|x+3| > 1 \quad \Rightarrow \quad |x+3| > 4 \qquad \text{series diverges}$$

radius of convergence for this power series is $R = 4$.

Now, let's get the interval of convergence. We'll get most (if not all) of the interval by solving the first inequality from above.

$$-4 < x+3 < 4$$

$$-7 < x < 1$$

The way to determine convergence at these points is to simply plug them into the original power series and see if the series converges or diverges using any test necessary.

$x = -7$:
In this case the series is,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-4)^n &= \sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-1)^n 4^n \\ &= \sum_{n=1}^{\infty} (-1)^n (-1)^n n \qquad\qquad (-1)^n(-1)^n = (-1)^{2n} = 1 \\ &= \sum_{n=1}^{\infty} n\end{aligned}$$

This series is divergent by the Divergence Test since $\lim_{n\to\infty} n = \infty \neq 0$.

$x = 1$:
In this case the series is,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(4)^n = \sum_{n=1}^{\infty} (-1)^n n$$

This series is also divergent by the Divergence Test since $\lim_{n\to\infty} (-1)^n n$ doesn't exist.

So, in this case the power series will not converge for either endpoint. The interval of convergence is then,

$$-7 < x < 1$$

Example 2 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$$

Solution
Let's jump right into the ratio test.

$$\begin{aligned}L &= \lim_{n\to\infty} \left| \frac{2^{n+1}(4x-8)^{n+1}}{n+1} \frac{n}{2^n(4x-8)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{2n(4x-8)}{n+1} \right| \\ &= |4x-8| \lim_{n\to\infty} \frac{2n}{n+1} \\ &= 2|4x-8|\end{aligned}$$

So we will get the following convergence/divergence information from this.

$$\begin{aligned}2|4x-8| &< 1 \qquad \text{series converges} \\ 2|4x-8| &> 1 \qquad \text{series diverges}\end{aligned}$$

We need to be careful here in determining the interval of convergence. The interval of convergence requires $\left|x-a\right| < R$ and $\left|x-a\right| > R$. In other words, we need to factor a 4 out of the absolute value bars in order to get the correct radius of convergence. Doing this gives,

$$8\left|x-2\right| < 1 \qquad \Rightarrow \qquad \left|x-2\right| < \frac{1}{8} \qquad \text{series converges}$$

$$8\left|x-2\right| > 1 \qquad \Rightarrow \qquad \left|x-2\right| > \frac{1}{8} \qquad \text{series diverges}$$

So, the radius of convergence for this power series is $R = \frac{1}{8}$.

Now, let's find the interval of convergence. Again, we'll first solve the inequality that gives convergence above.

$$-\frac{1}{8} < x - 2 < \frac{1}{8}$$
$$\frac{15}{8} < x < \frac{17}{8}$$

Now check the endpoints.

$x = \dfrac{15}{8}$:

The series here is,

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{15}{2} - 8\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(-\frac{1}{2}\right)^n$$
$$= \sum_{n=1}^{\infty} \frac{2^n}{n} \frac{(-1)^n}{2^n}$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

This is the alternating harmonic series and we know that it converges.

$x = \dfrac{17}{8}$:

The series here is,

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{17}{2} - 8\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{1}{2}\right)^n$$
$$= \sum_{n=1}^{\infty} \frac{2^n}{n} \frac{1}{2^n}$$
$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

The interval of convergence for this power series is

$$\frac{15}{8} \le x < \frac{17}{8}$$

Example 3 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=0}^{\infty} n!(2x+1)^n$$

Solution
We'll start this example with the ratio test as we have for the previous ones.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n+1)n!\,(2x+1)}{n!} \right| \\ &= |2x+1| \lim_{n\to\infty} (n+1) \end{aligned}$$

At this point we need to be careful. The limit is infinite, but there is that term with the x's in front of the limit. We'll have $L = \infty > 1$ provided $x \ne -\frac{1}{2}$.

Example 4 Determine the radius of convergence and interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} \frac{(x-6)^n}{n^n}$$

Solution
In this example the root test seems more appropriate. So,

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(x-6)^n}{n^n} \right|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \left| \frac{x-6}{n} \right| \\ &= |x-6| \lim_{n\to\infty} \frac{1}{n} \\ &= 0 \end{aligned}$$

So, since $L = 0 < 1$ regardless of the value of x this power series will converge for every x.

In these cases we say that the radius of convergence is $R = \infty$ and interval of convergence is $-\infty < x < \infty$.

C- ***Alternating Series.***

alternating series is any series, $\sum a_n$, for which the series terms can be written in one of the following two forms.

$$a_n = (-1)^n b_n \qquad b_n \ge 0$$
$$a_n = (-1)^{n+1} b_n \qquad b_n \ge 0$$

There are many other ways to deal with the alternating sign, but they can all be written as one of the two forms above. For instance,

$$(-1)^{n+2} = (-1)^n (-1)^2 = (-1)^n$$
$$(-1)^{n-1} = (-1)^{n+1} (-1)^{-2} = (-1)^{n+1}$$

Alternating Series Test

Suppose that we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \ge 0$ for all n. Then if,

1. $\lim\limits_{n\to\infty} b_n = 0$ and,
2. $\{b_n\}$ is a decreasing sequence

the series $\sum a_n$ is convergent.

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Solution

First, identify the b_n for the test.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \qquad b_n = \frac{1}{n}$$

Now, all that we need to do is run through the two conditions in the test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$
$$b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$$

Both conditions are met and so by the Alternating Series Test the series must converge.
The series from the previous example is sometimes called the **Alternating Harmonic Series**. Also, the $(-1)^{n+1}$ could be $(-1)^n$ or any other form of alternating sign and we'd still call it an Alternating Harmonic Series.

Example 2 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5}$$

Solution

First, identify the b_n for the test.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+5} = \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+5} \qquad \Rightarrow \qquad b_n = \frac{n^2}{n^2+5}$$

Let's check the conditions.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1 \neq 0$$

So, the divergence test requires us to compute the following limit.

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5}$$

This limit can be somewhat tricky to evaluate. For a second let's consider the following,

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} = \left(\lim_{n\to\infty} (-1)^n\right)\left(\lim_{n\to\infty} \frac{n^2}{n^2+5}\right)$$

So, let's start with,

$$\lim_{n\to\infty} \frac{(-1)^n n^2}{n^2+5} = \lim_{n\to\infty} \left[(-1)^n \frac{n^2}{n^2+5}\right]$$

Now, the second part of this clearly is going to 1 as $n \to \infty$ while the first part just alternates between 1 and -1. So, as $n \to \infty$ the terms are alternating between positive and negative values that are getting closer and closer to 1 and -1 respectively.

In order for limits to exist we know that the terms need to settle down to a single number and since these clearly don't this limit doesn't exist and so by the Divergence Test this series diverges.

Example 3 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+4}$$

Solution

$$b_n = \frac{\sqrt{n}}{n+4}$$

so let's check the conditions.

The first is easy enough to check.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{n+4} = 0$$

Let's start with the following function and its derivative.

$$f(x) = \frac{\sqrt{x}}{x+4} \qquad\qquad f'(x) = \frac{4-x}{2\sqrt{x}\,(x+4)^2}$$

Now, there are three critical points for this function, $x = -4$, $x = 0$, and $x = 4$. The first is outside the bound of our series so we won't need to worry about that one. Using the test points,

$$f'(1) = \frac{3}{50} \qquad\qquad f'(5) = -\frac{\sqrt{5}}{810}$$

and so we can see that the function in increasing on $0 \le x \le 4$ and decreasing on $x \ge 4$. Therefore, since $f(n) = b_n$ we know as well that the b_n are also increasing on $0 \le n \le 4$ and decreasing on $n \ge 4$.

The b_n are then eventually decreasing and so the second condition is met.

Both conditions are met and so by the Alternating Series Test the series must be converging.

Example 4 Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

Solution

The point of this problem is really just to acknowledge that it is in fact an alternating series. To see this we need to acknowledge that,

$$\cos(n\pi) = (-1)^n$$

and so the series is really,

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad \Rightarrow \qquad b_n = \frac{1}{\sqrt{n}}$$

Checking the two condition gives,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$$b_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = b_{n+1}$$

The two conditions of the test are met and so by the Alternating Series Test the series is convergent.

5- ***Comparison Test .***

Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then,

1. If $\sum b_n$ is convergent then so is $\sum a_n$.
2. If $\sum a_n$ is divergent then so is $\sum b_n$.

consider the following series.

$$\sum_{n=0}^{\infty} \frac{1}{3^n + n}$$

$$\frac{1}{3^n + n} < \frac{1}{3^n}$$

Now,

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

is a geometric series and we know that since $|r| = \left|\frac{1}{3}\right| < 1$ the series will converge and its value will be,

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

Now, if we go back to our original series and write down the partial sums we get,

$$s_n = \sum_{i=0}^{n} \frac{1}{3^i + i}$$

Since all the terms are positive adding a new term will only make the number larger and so the sequence of partial sums must be an increasing sequence.

$$s_n = \sum_{i=0}^{n} \frac{1}{3^i + i} < \sum_{i=0}^{n+1} \frac{1}{3^i + i} = s_{n+1}$$

Then since,

$$\frac{1}{3^n + n} < \frac{1}{3^n}$$

and because the terms in these two sequences are positive we can also say that,

$$s_n = \sum_{i=0}^{n} \frac{1}{3^i + i} < \sum_{i=0}^{n} \frac{1}{3^i} < \sum_{i=0}^{\infty} \frac{1}{3^n} = \frac{3}{2} \qquad \Rightarrow \qquad s_n < \frac{3}{2}$$

So, the sequence of partial sums of our series is a convergent sequence. This means that the series itself,

$$\sum_{n=0}^{\infty} \frac{1}{3^n + n}$$

is also convergent.

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2(n)}$$

Solution

Since the cosine term in the denominator doesn't get too large we can assume that the series terms will behave like,

$$\frac{n}{n^2} = \frac{1}{n}$$

Therefore,

$$\frac{n}{n^2 - \cos^2(n)} > \frac{n}{n^2} = \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges (it's harmonic or the p-series test) by the Comparison Test our original series must also diverge.

Example 2 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^4 + 5}$$

Solution

In this case the "+2" and the "+5" don't really add anything to the series and so the series terms should behave pretty much like

$$\frac{n^2}{n^4} = \frac{1}{n^2}$$

Let's take a look at the following series.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^4} &= \sum_{n=1}^{\infty} \frac{n^2}{n^4} + \sum_{n=1}^{\infty} \frac{2}{n^4} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n^4}\end{aligned}$$

As shown, we can write the series as a sum of two series and both of these series are convergent by the p-series test. Therefore, since each of these series are convergent we know that the sum,

$$\sum_{n=1}^{\infty} \frac{n^2+2}{n^4}$$

is also a convergent series. Recall that the sum of two convergent series will also be convergent.

Now, since the terms of this series are larger than the terms of the original series we know that the original series must also be convergent by the Comparison Test.

6- *Absolute Convergence.*

First, let's go back over the definition of absolute convergence.

Definition

A series $\sum a_n$ is called **absolutely convergent** if $\sum |a_n|$ is convergent. If $\sum a_n$ is convergent and $\sum |a_n|$ is divergent we call the series **conditionally convergent**.

We also have the following fact about absolute convergence.

Fact

If $\sum a_n$ is absolutely convergent then it is also convergent.

Example 1 Determine if each of the following series are absolute convergent, conditionally convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

This is the alternating harmonic series and we saw in the last section that it is a convergent series so we don't need to check that here. So, let's see if it is an absolutely convergent series. To do this we'll need to check the convergence of.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

that it is divergent.

Therefore, this series is not absolutely convergent. It is however conditionally convergent since the series itself does converge.

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{n^2}$

In this case let's just check absolute convergence first since if it's absolutely convergent we won't need to bother checking convergence as we will get that for free.

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+2}}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

This series is convergent by the p-series test and so the series is absolute convergent. Note that this does say as well that it's a convergent series.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^3}$

$$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^3}\right| = \sum_{n=1}^{\infty}\frac{|\sin n|}{n^3}$$

To do this we'll need to note that

$$-1 \le \sin n \le 1 \qquad \Rightarrow \qquad |\sin n| \le 1$$

and so we have,

$$\frac{|\sin n|}{n^3} \le \frac{1}{n^3}$$

Now we know that

$$\sum_{n=1}^{\infty}\frac{1}{n^3}$$

converges by the p-series test and so by the Comparison Test we also know that

$$\sum_{n=1}^{\infty}\frac{|\sin n|}{n^3}$$

converges.

Therefore the original series is absolutely convergent (and hence convergent).

7- ***Ratio Test.***

Ratio Test

Suppose we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

Then,

1. if $L < 1$ the series is absolutely convergent (and hence convergent).
2. if $L > 1$ the series is divergent.
3. if $L = 1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty}\frac{(-10)^n}{4^{2n+1}(n+1)}$$

Solution

With this first example let's be a little careful and make sure that we have everything down correctly. Here are the series terms a_n.

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Recall that to compute a_{n+1} all that we need to do is substitute $n+1$ for all the n's in a_n.

$$a_{n+1} = \frac{(-10)^{n+1}}{4^{2(n+1)+1}((n+1)+1)} = \frac{(-10)^{n+1}}{4^{2n+3}(n+2)}$$

Now, to define L we will use,

$$L = \lim_{n\to\infty}\left|a_{n+1}\cdot\frac{1}{a_n}\right|$$

since this will be a little easier when dealing with fractions as we've got here. So,

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)}\,\frac{4^{2n+1}(n+1)}{(-10)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{-10(n+1)}{4^2(n+2)}\right| \\ &= \frac{10}{16}\lim_{n\to\infty}\frac{n+1}{n+2} \\ &= \frac{10}{16}<1 \end{aligned}$$

So, $L<1$ and so by the Ratio Test the series converges absolutely and hence will converge.

Example 2 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty}\frac{n!}{5^n}$$

Solution

Now that we've worked one in detail we won't go into quite the detail with the rest of these. Here is the limit.

$$L = \lim_{n\to\infty}\left|\frac{(n+1)!}{5^{n+1}}\,\frac{5^n}{n!}\right| = \lim_{n\to\infty}\frac{(n+1)!}{5\;n!}$$

$$L = \lim_{n\to\infty}\frac{(n+1)\;n!}{5\;n!}$$

at which point we can cancel the $n!$ for the numerator an denominator to get,

$$L = \lim_{n\to\infty}\frac{(n+1)}{5} = \infty > 1$$

So, by the Ratio Test this series diverges.

Example 3 Determine if the following series is convergent or divergent.

$$\sum_{n=2}^{\infty}\frac{n^2}{(2n-1)!}$$

Solution

In this case be careful in dealing with the factorials.

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2(n+1)-1)!} \frac{(2n-1)!}{n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+1)!} \frac{(2n-1)!}{n^2} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n)(2n-1)!} \frac{(2n-1)!}{n^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n)(n^2)}$$

$$= 0 < 1$$

So, by the Ratio Test this series converges absolutely and so converges.

Example 4 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{9^n}{(-2)^{n+1} n}$$

Solution

Do not mistake this for a geometric series. The n in the denominator means that this isn't a geometric series. So, let's compute the limit.

$$L = \lim_{n\to\infty} \left| \frac{9^{n+1}}{(-2)^{n+2}(n+1)} \frac{(-2)^{n+1} n}{9^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{9n}{(-2)(n+1)} \right|$$

$$= \frac{9}{2} \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \frac{9}{2} > 1$$

Therefore, by the Ratio Test this series is divergent.

Example 5 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$$

Solution

Let's first get L.

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)^2+1} \frac{n^2+1}{(-1)^n} \right| = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1} = 1$$

So, as implied earlier we get $L=1$ which means the ratio test is no good for determining the convergence of this series. We will need to resort to another test for this series. This series is an alternating series and so let's check the two conditions from that test.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$$

$$b_n = \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} = b_{n+1}$$

The two conditions are met and so by the Alternating Series Test this series is convergent. We'll leave it to you to verify this series is also absolutely convergent.

Example 6 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{n+2}{2n+7}$$

Solution

Here's the limit.

$$L = \lim_{n\to\infty} \left| \frac{n+3}{2(n+1)+7} \frac{2n+7}{n+2} \right| = \lim_{n\to\infty} \frac{(n+3)(2n+7)}{(2n+9)(n+2)} = 1$$

Again, the ratio test tells us nothing here. We can however, quickly use the divergence test on this. In fact that probably should have been our first choice on this one anyway.

$$\lim_{n\to\infty} \frac{n+2}{2n+7} = \frac{1}{2} \neq 0$$

By the Divergence Test this series is divergent.

8- <u>Root Test.</u>

Root Test

Suppose that we have the series $\sum a_n$. Define,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Then,

4. if $L<1$ the series is absolutely convergent (and hence convergent).
5. if $L>1$ the series is divergent.
6. if $L=1$ the series may be divergent, conditionally convergent, or absolutely convergent.

Fact

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

Example 1 Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$$

Solution

There really isn't much to these problems other than computing the limit and then using the root test. Here is the limit for this problem.

$$L = \lim_{n\to\infty} \left| \frac{n^n}{3^{1+2n}} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3^{\frac{1}{n}+2}} = \frac{\infty}{3^2} = \infty > 1$$

So, by the Root Test this series is divergent.

Example 2 Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \left(\frac{5n-3n^3}{7n^3+2} \right)^n$$

Solution

Again, there isn't too much to this series.

$$L = \lim_{n\to\infty} \left| \left(\frac{5n-3n^3}{7n^3+2} \right)^n \right|^{\frac{1}{n}} = \lim_{n\to\infty} \left| \frac{5n-3n^3}{7n^3+2} \right| = \left| \frac{-3}{7} \right| = \frac{3}{7} < 1$$

Therefore, by the Root Test this series converges absolutely and hence converges.

Example 3 Determine if the following series is convergent or divergent.

$$\sum_{n=3}^{\infty} \frac{(-12)^n}{n}$$

Solution

Here's the limit for this series.

$$L = \lim_{n\to\infty} \left| \frac{(-12)^n}{n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{12}{n^{\frac{1}{n}}} = \frac{12}{1} = 12 > 1$$

After using the fact from above we can see that the Root Test tells us that this series is divergent.

9- ***Taylor Series.***

So, for the time being, let's make two assumptions. First, let's assume that the function $f(x)$ does in fact have a power series representation about $x = a$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

Next, we will need to assume that the function, $f(x)$, has derivatives of every order and that we can in fact find them all.

Now that we've assumed that a power series representation exists we need to determine what the coefficients, c_n, are. This is easier than it might at first appear to be. Let's first just evaluate everything at $x = a$. This gives,

$$f(a) = c_0$$

However, if we take the derivative of the function (and its power series) then plug in $x = a$ we get,

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$
$$f'(a) = c_1$$

and we now know c_1.

Let's continue with this idea and find the second derivative.

$$f''(x) = 2c_2 + 3(2)c_3(x-a) + 4(3)c_4(x-a)^2 + \cdots$$
$$f''(a) = 2c_2$$

So, it looks like,

$$c_2 = \frac{f''(a)}{2}$$

Using the third derivative gives,

$$f'''(x) = 3(2)c_3 + 4(3)(2)c_4(x-a) + \cdots$$

$$f'''(a) = 3(2)c_3 \qquad \Rightarrow \qquad c_3 = \frac{f'''(a)}{3(2)}$$

Using the fourth derivative gives,

$$f^{(4)}(x) = 4(3)(2)c_4 + 5(4)(3)(2)c_5(x-a)\cdots$$

$$f^{(4)}(a) = 4(3)(2)c_4 \qquad \Rightarrow \qquad c_4 = \frac{f^{(4)}(a)}{4(3)(2)}$$

Hopefully by this time you've seen the pattern here. It looks like, in general, we've got the following formula for the coefficients.

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This even works for *n=0* if you recall that $0! = 1$ and define $f^{(0)}(x) = f(x)$.

So, provided a power series representation for the function $f(x)$ about $x = a$ exists the **Taylor Series for $f(x)$ about $x = a$** is,

Taylor Series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

If we use $a = 0$, so we are talking about the Taylor Series about $x = 0$, we call the series a **Maclaurin Series** for $f(x)$ or,

Maclaurin Series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \end{aligned}$$

To determine a condition that must be true in order for a Taylor series to exist for a function let's first define the **n$^{\text{th}}$ degree Taylor polynomial of $f(x)$** as,

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Notice as well that for the full Taylor Series,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Next, the **remainder** is defined to be,

$$R_n(x) = f(x) - T_n(x)$$

So, the remainder is really just the *error* between the function $f(x)$ and the n$^{\text{th}}$ degree Taylor polynomial for a given *n*.

With this definition note that we can then write the function as,

$$f(x) = T_n(x) + R_n(x)$$

Theorem

Suppose that $f(x) = T_n(x) + R_n(x)$. Then if,

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$ then,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

on $|x-a| < R$.

Example 1 Find the Taylor Series for $f(x) = \mathbf{e}^x$ about $x = 0$.

Solution

To get a formula for $f^{(n)}(0)$ all we need to do is recognize that,

$$f^{(n)}(x) = \mathbf{e}^x \qquad\qquad n = 0,1,2,3,\ldots$$

and so,

$$f^{(n)}(0) = \mathbf{e}^0 = 1 \qquad\qquad n = 0,1,2,3,\ldots$$

Therefore, the Taylor series for $f(x) = \mathbf{e}^x$ about $x=0$ is,

$$\mathbf{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example 2 Find the Taylor Series for $f(x) = \mathbf{e}^{-x}$ about $x = 0$.

Solution

Solution 1

As with the first example we'll need to get a formula for $f^{(n)}(0)$. However, unlike the first one we've got a little more work to do. Let's first take some derivatives and evaluate them at $x=0$.

$$\begin{array}{ll}
f^{(0)}(x) = \mathbf{e}^{-x} & f^{(0)}(0) = 1 \\
f^{(1)}(x) = -\mathbf{e}^{-x} & f^{(1)}(0) = -1 \\
f^{(2)}(x) = \mathbf{e}^{-x} & f^{(2)}(0) = 1 \\
f^{(3)}(x) = -\mathbf{e}^{-x} & f^{(3)}(0) = -1 \\
\quad\vdots & \quad\vdots \\
f^{(n)}(x) = (-1)^n \mathbf{e}^{-x} & f^{(n)}(0) = (-1)^n \qquad n = 0,1,2,3
\end{array}$$

So, in this case we've got general formulas so all we need to do is plug these into the Taylor Series formula and be done with the problem.

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

Solution 2

So, all we need to do is replace the x in the Taylor Series that we found in the first example with "$-x$".

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$$

This is a much shorter method of arriving at the same answer so don't forget about using previously computed series where possible (and allowed of course).

Example 3 Find the Taylor Series for $f(x) = x^4 \mathbf{e}^{-3x^2}$ about $x = 0$.

Solution

For this example we will take advantage of the fact that we already have a Taylor Series for $\mathbf{e}^x$ about $x = 0$. In this example, unlike the previous example, doing this directly would be significantly longer and more difficult.

$$\begin{aligned} x^4 \mathbf{e}^{-3x^2} &= x^4 \sum_{n=0}^{\infty} \frac{(-3x^2)^n}{n!} \\ &= x^4 \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+4}}{n!} \end{aligned}$$

To this point we've only looked at Taylor Series about $x = 0$ (also known as Maclaurin Series) so let's take a look at a Taylor Series that isn't about $x = 0$.

Example 4 Find the Taylor Series for $f(x) = \mathbf{e}^{-x}$ about $x = -4$.

Solution

Finding a general formula for $f^{(n)}(-4)$ is fairly simple.

$$f^{(n)}(x) = (-1)^n \mathbf{e}^{-x} \qquad\qquad f^{(n)}(-4) = (-1)^n \mathbf{e}^4$$

The Taylor Series is then,

$$\mathbf{e}^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n \mathbf{e}^4}{n!} (x+4)^n$$

Example 5 Find the Taylor Series for $f(x) = \cos(x)$ about $x = 0$.

Solution
First we'll need to take some derivatives of the function and evaluate them at *x=0*.

$$\begin{aligned}
f^{(0)}(x) &= \cos x & f^{(0)}(0) &= 1 \\
f^{(1)}(x) &= -\sin x & f^{(1)}(0) &= 0 \\
f^{(2)}(x) &= -\cos x & f^{(2)}(0) &= -1 \\
f^{(3)}(x) &= \sin x & f^{(3)}(0) &= 0 \\
f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \\
f^{(5)}(x) &= -\sin x & f^{(5)}(0) &= 0 \\
f^{(6)}(x) &= -\cos x & f^{(6)}(0) &= -1
\end{aligned}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots$$

$$= \underbrace{1}_{n=0} + \underbrace{0}_{n=1} \underbrace{-\frac{1}{2!}x^2}_{n=2} + \underbrace{0}_{n=3} \underbrace{+\frac{1}{4!}x^4}_{n=4} + \underbrace{0}_{n=5} \underbrace{-\frac{1}{6!}x^6}_{n=6} + \cdots$$

$$\cos x = \underbrace{1}_{n=0} \underbrace{-\frac{1}{2!}x^2}_{n=1} \underbrace{+\frac{1}{4!}x^4}_{n=2} \underbrace{-\frac{1}{6!}x^6}_{n=3} + \cdots$$

$$\cos x = \underbrace{1}_{n=0} \underbrace{-\frac{1}{2!}x^2}_{n=1} \underbrace{+\frac{1}{4!}x^4}_{n=2} \underbrace{-\frac{1}{6!}x^6}_{n=3} + \cdots$$

By renumbering the terms as we did we can actually come up with a general formula for the Taylor Series and here it is,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Example 6 Find the Taylor Series for $f(x) = \sin(x)$ about $x = 0$.

Solution
As with the last example we'll start off in the same manner.

$$\begin{aligned} f^{(0)}(x) &= \sin x & f^{(0)}(0) &= 0 \\ f^{(1)}(x) &= \cos x & f^{(1)}(0) &= 1 \\ f^{(2)}(x) &= -\sin x & f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= -\cos x & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= \cos x & f^{(5)}(0) &= 1 \\ f^{(6)}(x) &= -\sin x & f^{(6)}(0) &= 0 \end{aligned}$$

So, we get a similar pattern for this one. Let's plug the numbers into the Taylor Series.

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \end{aligned}$$

So renumbering the terms as we did in the previous example we get the following Taylor Series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We really need to work another example or two in which $f(x)$ isn't about $x = 0$.

Example 7 Find the Taylor Series for $f(x) = \ln(x)$ about $x = 2$.

Solution
Here are the first few derivatives and the evaluations.

$$f^{(0)}(x) = \ln(x) \qquad f^{(0)}(2) = \ln 2$$

$$f^{(1)}(x) = \frac{1}{x} \qquad f^{(1)}(2) = \frac{1}{2}$$

$$f^{(2)}(x) = -\frac{1}{x^2} \qquad f^{(2)}(2) = -\frac{1}{2^2}$$

$$f^{(3)}(x) = \frac{2}{x^3} \qquad f^{(3)}(2) = \frac{2}{2^3}$$

$$f^{(4)}(x) = -\frac{2(3)}{x^4} \qquad f^{(4)}(2) = -\frac{2(3)}{2^4}$$

$$f^{(5)}(x) = \frac{2(3)(4)}{x^5} \qquad f^{(5)}(2) = \frac{2(3)(4)}{2^5}$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n} \qquad f^{(n)}(2) = \frac{(-1)^{n+1}(n-1)!}{2^n} \qquad n = 1, 2, 3, \ldots$$

In order to plug this into the Taylor Series formula we'll need to strip out the $n = 0$ term first.

$$\begin{aligned}
\ln(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= f(2) + \sum_{n=1}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!\ 2^n}(x-2)^n \\
&= \ln(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,2^n}(x-2)^n
\end{aligned}$$

Example 8 Find the Taylor Series for $f(x) = \frac{1}{x^2}$ about $x = -1$.

Solution

Again, here are the derivatives and evaluations.

$$f^{(0)}(x) = \frac{1}{x^2} \qquad f^{(0)}(-1) = \frac{1}{(-1)^2} = 1$$

$$f^{(1)}(x) = -\frac{2}{x^3} \qquad f^{(1)}(-1) = -\frac{2}{(-1)^3} = 2$$

$$f^{(2)}(x) = \frac{2(3)}{x^4} \qquad f^{(2)}(-1) = \frac{2(3)}{(-1)^4} = 2(3)$$

$$f^{(3)}(x) = -\frac{2(3)(4)}{x^5} \qquad f^{(3)}(-1) = -\frac{2(3)(4)}{(-1)^5} = 2(3)(4)$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$f^{(n)}(x) = \frac{(-1)^n (n+1)!}{x^{n+2}} \qquad f^{(n)}(-1) = \frac{(-1)^n (n+1)!}{(-1)^{n+2}} = (n+1)!$$

Here is the Taylor Series for this function.

$$\begin{aligned}\frac{1}{x^2} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n \\ &= \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}(x+1)^n \\ &= \sum_{n=0}^{\infty} (n+1)(x+1)^n\end{aligned}$$

Example 9 Find the Taylor Series for $f(x) = x^3 - 10x^2 + 6$ about $x = 3$.

Solution

Here are the derivatives for this problem.

$$\begin{aligned} f^{(0)}(x) &= x^3 - 10x^2 + 6 & f^{(0)}(3) &= -57 \\ f^{(1)}(x) &= 3x^2 - 20x & f^{(1)}(3) &= -33 \\ f^{(2)}(x) &= 6x - 20 & f^{(2)}(3) &= -2 \\ f^{(3)}(x) &= 6 & f^{(3)}(3) &= 6 \\ f^{(n)}(x) &= 0 & f^{(4)}(3) &= 0 \qquad n \geq 4 \end{aligned}$$

This Taylor series will terminate after $n = 3$. This will always happen when we are finding the Taylor Series of a polynomial. Here is the Taylor Series for this one.

$$\begin{aligned} x^3 - 10x^2 + 6 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n \\ &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + 0 \\ &= -57 - 33(x-3) - (x-3)^2 + (x-3)^3 \end{aligned}$$

10- Problems.

A- Sequences.

For problems 1 & 2 list the first 5 terms of the sequence.

1. $\left\{\dfrac{4n}{n^2-7}\right\}_{n=0}^{\infty}$

2. $\left\{\dfrac{(-1)^{n+1}}{2n+(-3)^n}\right\}_{n=2}^{\infty}$

For problems 3 – 6 determine if the given sequence converges or diverges. If it converges what is its limit?

3. $\left\{\dfrac{n^2-7n+3}{1+10n-4n^2}\right\}_{n=3}^{\infty}$

4. $\left\{\dfrac{(-1)^{n-2}n^2}{4+n^3}\right\}_{n=0}^{\infty}$

5. $\left\{\dfrac{\mathbf{e}^{5n}}{3-\mathbf{e}^{2n}}\right\}_{n=1}^{\infty}$

6. $\left\{\dfrac{\ln(n+2)}{\ln(1+4n)}\right\}_{n=1}^{\infty}$

For each of the following problems determine if the sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

1. $\left\{\dfrac{1}{4n}\right\}_{n=1}^{\infty}$

2. $\left\{n(-1)^{n+2}\right\}_{n=0}^{\infty}$

3. $\left\{3^{-n}\right\}_{n=0}^{\infty}$

4. $\left\{\dfrac{2n^2-1}{n}\right\}_{n=2}^{\infty}$

5. $\left\{\dfrac{4-n}{2n+3}\right\}_{n=1}^{\infty}$

B- Series.

For problems 1 & 2 compute the first 3 terms in the sequence of partial sums for the given series.

1. $\sum_{n=1}^{\infty} n\,2^n$

2. $\sum_{n=3}^{\infty} \frac{2n}{n+2}$

For problems 3 & 4 assume that the n^{th} term in the sequence of partial sums for the series $\sum_{n=0}^{\infty} a_n$ is given below. Determine if the series $\sum_{n=0}^{\infty} a_n$ is convergent or divergent. If the series is convergent determine the value of the series.

3. $s_n = \frac{5+8n^2}{2n-7n^2}$

4. $s_n = \frac{n^2}{5+2n}$

For problems 5 & 6 show that the series is divergent.

5. $\sum_{n=0}^{\infty} \frac{3n\,\mathbf{e}^n}{n^2+1}$

6. $\sum_{n=5}^{\infty} \frac{6+8n+9n^2}{3+2n+n^2}$

For each of the following series determine if the series converges or diverges. If the series converges give its value.

1. $\sum_{n=0}^{\infty} 3^{2+n}\,2^{1-3n}$

2. $\sum_{n=1}^{\infty} \frac{5}{6n}$

3. $\sum_{n=1}^{\infty} \frac{(-6)^{3-n}}{8^{2-n}}$

4. $\sum_{n=1}^{\infty} \frac{3}{n^2+7n+12}$

C- Comparison Test.

For each of the following series determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+1\right)^2$

2. $\sum_{n=4}^{\infty}\frac{n^2}{n^3-3}$

3. $\sum_{n=2}^{\infty}\frac{7}{n(n+1)}$

4. $\sum_{n=7}^{\infty}\frac{4}{n^2-2n-3}$

5. $\sum_{n=2}^{\infty}\frac{n-1}{\sqrt{n^6+1}}$

6. $\sum_{n=1}^{\infty}\frac{2n^3+7}{n^4\sin^2(n)}$

D- Absolute Convergence.

For each of the following series determine if they are absolutely convergent, conditionally convergent or divergent.

1. $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^3+1}$

2. $\sum_{n=1}^{\infty}\frac{(-1)^{n-3}}{\sqrt{n}}$

3. $\sum_{n=3}^{\infty}\frac{(-1)^{n+1}(n+1)}{n^3+1}$

E- ***Ratio Test.***

For each of the following series determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1}$

2. $\sum_{n=0}^{\infty} \frac{(2n)!}{5n+1}$

3. $\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$

4. $\sum_{n=3}^{\infty} \frac{\mathbf{e}^{4n}}{(n-2)!}$

F- ***Root Test.***

For each of the following series determine if the series converges or diverges.

1. $\sum_{n=1}^{\infty} \left(\frac{3n+1}{4-2n}\right)^{2n}$

2. $\sum_{n=0}^{\infty} \frac{n^{1-3n}}{4^{2n}}$

3. $\sum_{n=4}^{\infty} \frac{(-5)^{1+2n}}{2^{5n-3}}$

G- ***Power Series***.

For each of the following power series determine the interval and radius of convergence.

1. $\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}(n^2+1)}(4x-12)^n$

2. $\sum_{n=0}^{\infty} \frac{n^{2n+1}}{4^{3n}}(2x+17)^n$

3. $\sum_{n=0}^{\infty} \frac{n+1}{(2n+1)!}(x-2)^n$

4. $\sum_{n=0}^{\infty} \frac{4^{1+2n}}{5^{n+1}}(x+3)^n$

5. $\sum_{n=0}^{\infty} \frac{6^n}{n}(4x-1)^{n-1}$

H- ***Taylor Series***.

For problems 1 & 2 use one of the Taylor Series derived in the notes to determine the Taylor Series for the given function.

1. $f(x) = \cos(4x)$ about $x = 0$

2. $f(x) = x^6 \mathbf{e}^{2x^3}$ about $x = 0$

For problem 3 – 6 find the Taylor Series for each of the following functions.

3. $f(x) = \mathbf{e}^{-6x}$ about $x = -4$

4. $f(x) = \ln(3 + 4x)$ about $x = 0$

5. $f(x) = \dfrac{7}{x^4}$ about $x = -3$

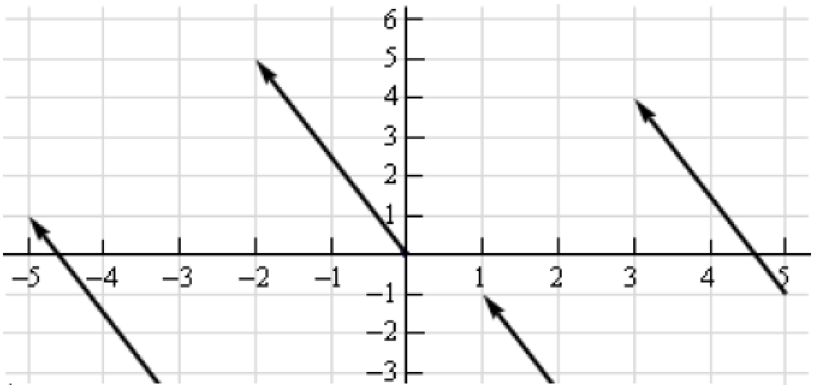
Vectors

1- Vectors – The Basics

The length of the line

segment is the magnitude of the vector and the direction of the line segment is the direction of the vector. However, because vectors don't impart any information about where the quantity is applied any directed line segment with the same length and direction will represent the same vector.

Consider the sketch below.



A representation of the vector $\vec{v} = \langle a_1, a_2 \rangle$ in two dimensional space is any directed line segment, $\overrightarrow{AB}$, from the point $A = (x, y)$ to the point $B = (x + a_1, y + a_2)$. Likewise a representation of the vector $\vec{v} = \langle a_1, a_2, a_3 \rangle$ in three dimensional space is any directed line segment, $\overrightarrow{AB}$, from the point $A = (x, y, z)$ to the point $B = (x + a_1, y + a_2, z + a_3)$.

Next we need to discuss briefly how to generate a vector given the initial and final points of the representation. Given the two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ the vector with the representation $\overrightarrow{AB}$ is,

$$\vec{v} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3 \rangle$$

Note that we have to be very careful with direction here. The vector above is the vector that starts at A and ends at B. The vector that starts at B and ends at A, *i.e.* with representation $\overrightarrow{BA}$ is,

$$\vec{w} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

When determining the vector between two

points we always subtract the initial point from the terminal point.

Example 1 Give the vector for each of the following.

(a) The vector from $(2,-7,0)$ to $(1,-3,-5)$.

(b) The vector from $(1,-3,-5)$ to $(2,-7,0)$.

(c) The position vector for $(-90,4)$

Solution

(a) Remember that to construct this vector we subtract coordinates of the starting point from the ending point.

$$\langle 1-2,-3-(-7),-5-0\rangle = \langle -1,4,-5\rangle$$

(b) Same thing here.

$$\langle 2-1,-7-(-3),0-(-5)\rangle = \langle 1,-4,5\rangle$$

Notice that the only difference between the first two is the signs are all opposite. This difference is important as it is this difference that tells us that the two vectors point in opposite directions.

(c) Not much to this one other than acknowledging that the position vector of a point is nothing more than a vector with the point's coordinates as its components.

$$\langle -90,4\rangle$$

Magnitude

The **magnitude**, or **length**, of the vector $\vec{v} = \langle a_1, a_2, a_3\rangle$ is given by,

$$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example 2 Determine the magnitude of each of the following vectors.

(a) $\vec{a} = \langle 3,-5,10\rangle$

(b) $\vec{u} = \left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right\rangle$

(c) $\vec{w} = \langle 0,0\rangle$

(d) $\vec{i} = \langle 1,0,0\rangle$

Solution

There isn't too much to these other than plug into the formula.

(a) $\|\vec{a}\| = \sqrt{9+25+100} = \sqrt{134}$

(b) $\|\vec{u}\| = \sqrt{\frac{1}{5}+\frac{4}{5}} = \sqrt{1} = 1$

(c) $\|\vec{w}\| = \sqrt{0+0} = 0$

(d) $\|\vec{i}\| = \sqrt{1+0+0} = 1$

We also have the following fact about the magnitude.

$$\text{If } \|\vec{a}\| = 0 \text{ then } \vec{a} = \vec{0}$$

This should make sense. Because we square all the components the only way we can get zero out of the formula was for the components to be zero in the first place.

Unit Vector

Any vector with magnitude of 1, *i.e.* $\|\vec{u}\| = 1$, is called a **unit vector**.

Zero Vector

The vector $\vec{w} = \langle 0, 0 \rangle$ that we saw in the first example is called a zero vector since its components are all zero. Zero vectors are often denoted by $\vec{0}$. Be careful to distinguish 0 (the number) from $\vec{0}$ (the vector). The number 0 denotes the origin in space, while the vector $\vec{0}$ denotes a vector that has no magnitude or direction.

Standard Basis Vectors

The fourth vector from the second example, $\vec{i} = \langle 1, 0, 0 \rangle$, is called a **standard basis vector**. In three dimensional space there are three standard basis vectors,

$$\vec{i} = \langle 1, 0, 0 \rangle \qquad \vec{j} = \langle 0, 1, 0 \rangle \qquad \vec{k} = \langle 0, 0, 1 \rangle$$

In two dimensional space there are two standard basis vectors,

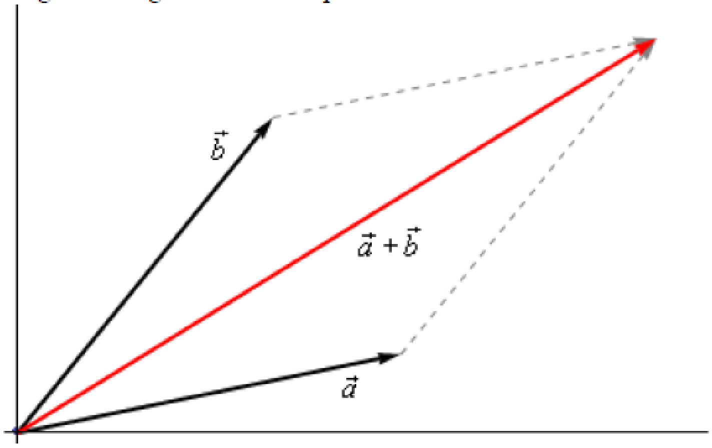
$$\vec{i} = \langle 1, 0 \rangle \qquad \vec{j} = \langle 0, 1 \rangle$$

2- <u>Vector Arithmetic</u>

We'll start with **addition** of two vectors. So, given the vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the addition of the two vectors is given by the following formula.

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

The following figure gives the geometric interpretation of the addition of two vectors.

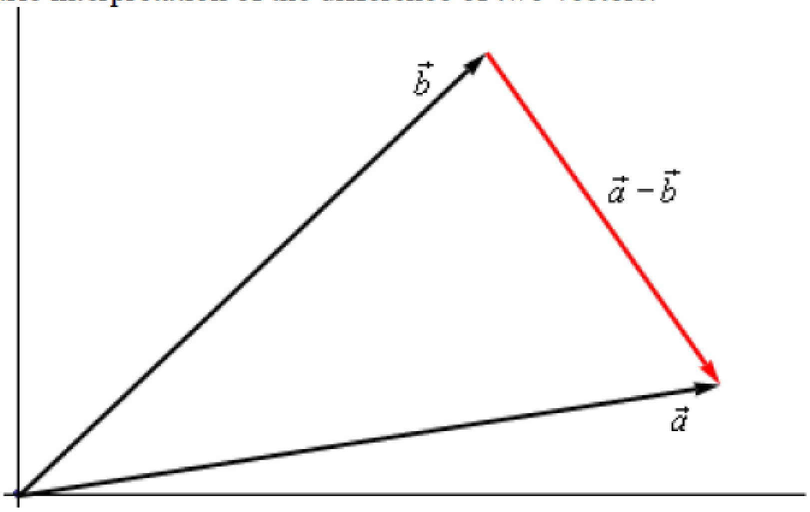


This is sometimes called the **parallelogram law** or **triangle law**.

Computationally, **subtraction** is very similar. Given the vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the difference of the two vectors is given by,

$$\vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

Here is the geometric interpretation of the difference of two vectors.



The next arithmetic operation that we want to look at is **scalar multiplication**. Given the vector $\vec{a}=\langle a_1, a_2, a_3\rangle$ and any number c the scalar multiplication is,

$$c\vec{a}=\langle ca_1, ca_2, ca_3\rangle$$

So, we multiply all the components by the constant c. To see the geometric interpretation of scalar multiplication let's take a look at an example.

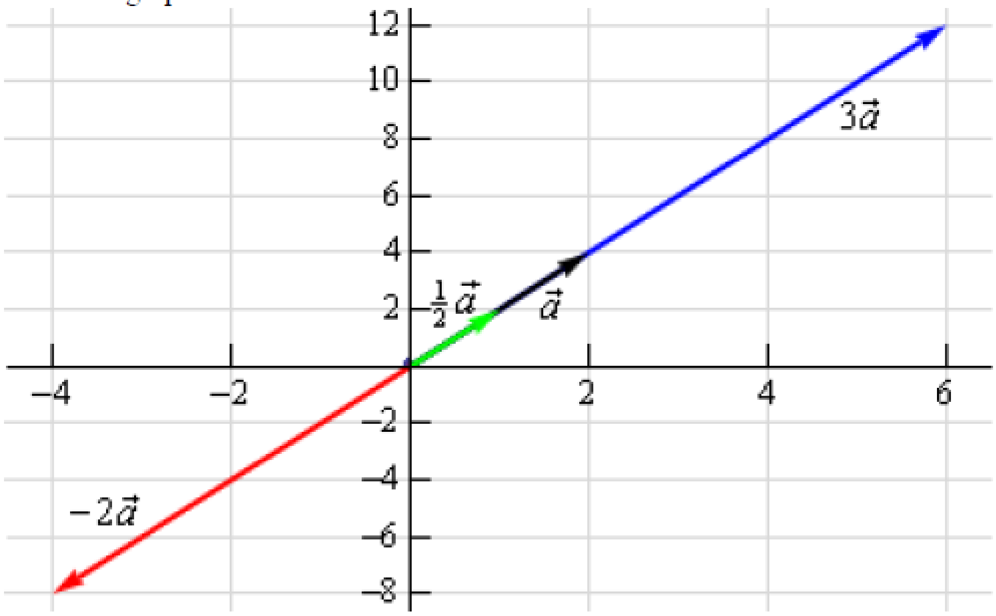
Example 1 For the vector $\vec{a}=\langle 2,4\rangle$ compute $3\vec{a}$, $\frac{1}{2}\vec{a}$ and $-2\vec{a}$. Graph all four vectors on the same axis system.

Solution

Here are the three scalar multiplications.

$$3\vec{a}=\langle 6,12\rangle \qquad \frac{1}{2}\vec{a}=\langle 1,2\rangle \qquad -2\vec{a}=\langle -4,-8\rangle$$

Here is the graph for each of these vectors.



The first is parallel vectors. This is a concept that we will see quite a bit over the next couple of sections. Two vectors are parallel if they have the same direction or are in exactly opposite directions. Now, recall again the geometric interpretation of scalar multiplication. When we performed scalar multiplication we generated new vectors that were parallel to the original vectors (and each other for that matter).

So, let's suppose that $\vec{a}$ and $\vec{b}$ are parallel vectors. If they are parallel then there must be a number c so that,

$$\vec{a} = c\vec{b}$$

So, two vectors are parallel if one is a scalar multiple of the other.

Example 2 Determine if the sets of vectors are parallel or not.

(a) $\vec{a} = \langle 2, -4, 1 \rangle$, $\vec{b} = \langle -6, 12, -3 \rangle$

(b) $\vec{a} = \langle 4, 10 \rangle$, $\vec{b} = \langle 2, -9 \rangle$

Solution

(a) These two vectors are parallel since $\vec{b} = -3\vec{a}$

(b) These two vectors aren't parallel. This can be seen by noticing that $4\left(\frac{1}{2}\right) = 2$ and yet $10\left(\frac{1}{2}\right) = 5 \neq -9$. In other words we can't make $\vec{a}$ be a scalar multiple of $\vec{b}$.

The next application is best seen in an example.

Example 3 Find a unit vector that points in the same direction as $\vec{w} = \langle -5, 2, 1 \rangle$.

Solution

Okay, what we're asking for is a new parallel vector (points in the same direction) that happens to be a unit vector. We can do this with a scalar multiplication since all scalar multiplication does is change the length of the original vector (along with possibly flipping the direction to the opposite direction).

Here's what we'll do. First let's determine the magnitude of $\vec{w}$.

$$\|\vec{w}\| = \sqrt{25 + 4 + 1} = \sqrt{30}$$

Now, let's form the following new vector,

$$\vec{u} = \frac{1}{\|\vec{w}\|}\vec{w} = \frac{1}{\sqrt{30}}\langle -5, 2, 1 \rangle = \left\langle -\frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}} \right\rangle$$

The claim is that this is a unit vector. That's easy enough to check

$$\|\vec{u}\| = \sqrt{\frac{25}{30} + \frac{4}{30} + \frac{1}{30}} = \sqrt{\frac{30}{30}} = 1$$

This vector also points in the same direction as $\vec{w}$ since it is only a scalar multiple of $\vec{w}$ and we used a positive multiple.

So, in general, given a vector $\vec{w}$, $\vec{u} = \dfrac{\vec{w}}{\|\vec{w}\|}$ will be a unit vector that points in the same direction as $\vec{w}$.

Properties

If $\vec{v}$, $\vec{w}$ and $\vec{u}$ are vectors (each with the same number of components) and a and b are two numbers then we have the following properties.

$$\vec{v} + \vec{w} = \vec{w} + \vec{v} \qquad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

$$\vec{v} + \vec{0} = \vec{v} \qquad 1\vec{v} = \vec{v}$$

$$a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w} \qquad (a+b)\vec{v} = a\vec{v} + b\vec{v}$$

3- ***Dot Product***

The next topic for discussion is that of the dot product. Let's jump right into the definition of the dot product. Given the two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ the dot product is,

$$\vec{a} \bullet \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1}$$

Sometimes the dot product is called the **scalar product**. The dot product is also an example of an inner product and so on occasion you may hear it called an inner product.

Example 1 Compute the dot product for each of the following.

(a) $\vec{v} = 5\vec{i} - 8\vec{j}$, $\vec{w} = \vec{i} + 2\vec{j}$

(b) $\vec{a} = \langle 0, 3, -7 \rangle$, $\vec{b} = \langle 2, 3, 1 \rangle$

Solution

Not much to do with these other than use the formula.

(a) $\vec{v} \bullet \vec{w} = 5 - 16 = -11$

(b) $\vec{a} \bullet \vec{b} = 0 + 9 - 7 = 2$

Here are some properties of the dot product.

Properties

$$\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w} \qquad (c\vec{v}) \bullet \vec{w} = \vec{v} \bullet (c\vec{w}) = c(\vec{v} \bullet \vec{w})$$

$$\vec{v} \bullet \vec{w} = \vec{w} \bullet \vec{v} \qquad \vec{v} \bullet \vec{0} = 0$$

$$\vec{v} \bullet \vec{v} = \|\vec{v}\|^2 \qquad \text{If } \vec{v} \bullet \vec{v} = 0 \text{ then } \vec{v} = \vec{0}$$

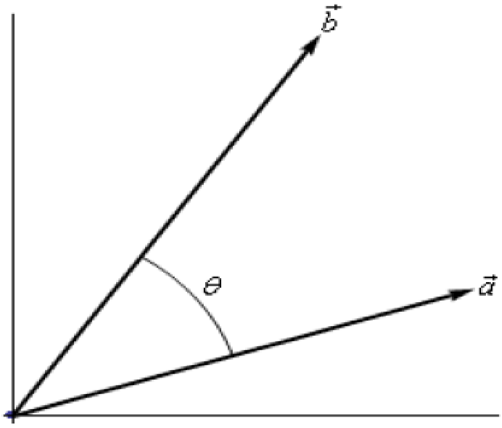
The proofs of these properties are mostly "computational" proofs and so we're only going to do a couple of them and leave the rest to you to prove.

Proof of $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

We'll start with the three vectors, $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$, $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ and $\vec{w} = \langle w_1, w_2, \ldots, w_n \rangle$ and yes we did mean for these to each have n components. The theorem works for general vectors so we may as well do the proof for general vectors.

$$\begin{aligned}
\vec{u}\bullet(\vec{v}+\vec{w}) &= \langle u_1, u_2, \ldots, u_n\rangle\bullet(\langle v_1, v_2, \ldots, v_n\rangle + \langle w_1, w_2, \ldots, w_n\rangle) \\
&= \langle u_1, u_2, \ldots, u_n\rangle\bullet\langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n\rangle \\
&= \langle u_1(v_1 + w_1), u_2(v_2 + w_2), \ldots, u_n(v_n + w_n)\rangle \\
&= \langle u_1 v_1 + u_1 w_1, u_2 v_2 + u_2 w_2, \ldots, u_n v_n + u_n w_n\rangle \\
&= \langle u_1 v_1, u_2 v_2, \ldots, u_n v_n\rangle + \langle u_1 w_1, u_2 w_2, \ldots, u_n w_n\rangle \\
&= \langle u_1, u_2, \ldots, u_n\rangle\bullet\langle v_1, v_2, \ldots, v_n\rangle + \langle u_1, u_2, \ldots, u_n\rangle\bullet\langle w_1, w_2, \ldots, w_n\rangle \\
&= \vec{u}\bullet\vec{v} + \vec{u}\bullet\vec{w}
\end{aligned}$$

There is also a nice geometric interpretation to the dot product. First suppose that θ is the angle between $\vec{a}$ and $\vec{b}$ such that $0 \le \theta \le \pi$ as shown in the image below.



We can then have the following theorem.

4- Applications of Dot Products

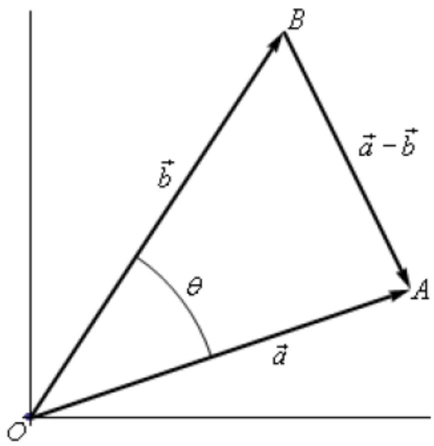
A. Find the angle between two vectors.

Theorem

$$\vec{a}\bullet\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta \qquad (2)$$

Proof

Let's give a modified version of the sketch above.



The three vectors above form the triangle AOB and note that the length of each side is nothing more than the magnitude of the vector forming that side.

The Law of Cosines tells us that,

$$\left\|\vec{a}-\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 - 2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$

Also using the properties of dot products we can write the left side as,

$$\begin{aligned}\left\|\vec{a}-\vec{b}\right\|^2 &= \left(\vec{a}-\vec{b}\right)\bullet\left(\vec{a}-\vec{b}\right)\\ &= \vec{a}\bullet\vec{a} - \vec{a}\bullet\vec{b} - \vec{b}\bullet\vec{a} + \vec{b}\bullet\vec{b}\\ &= \left\|\vec{a}\right\|^2 - 2\vec{a}\bullet\vec{b} + \left\|\vec{b}\right\|^2\end{aligned}$$

Our original equation is then,

$$\begin{aligned}\left\|\vec{a}-\vec{b}\right\|^2 &= \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 - 2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta\\ \left\|\vec{a}\right\|^2 - 2\vec{a}\bullet\vec{b} + \left\|\vec{b}\right\|^2 &= \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 - 2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta\\ -2\vec{a}\bullet\vec{b} &= -2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta\\ \vec{a}\bullet\vec{b} &= \left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta\end{aligned}$$

■

Example 2 Determine the angle between $\vec{a} = \langle 3, -4, -1\rangle$ and $\vec{b} = \langle 0, 5, 2\rangle$.

Solution

We will need the dot product as well as the magnitudes of each vector.

$$\vec{a}\bullet\vec{b} = -22 \qquad \left\|\vec{a}\right\| = \sqrt{26} \qquad \left\|\vec{b}\right\| = \sqrt{29}$$

The angle is then,

$$\cos\theta = \frac{\vec{a}\bullet\vec{b}}{\left\|\vec{a}\right\|\left\|\vec{b}\right\|} = \frac{-22}{\sqrt{26}\sqrt{29}} = -0.8011927$$

$$\theta = \cos^{-1}\left(-0.8011927\right) = 2.5 \text{ radians} = 143.24 \text{ degrees}$$

B. <u>*Determine parallel and orthogonal of vectors.*</u>

The dot product gives us a very nice method for determining if two vectors are perpendicular and it will give another method for determining when two vectors are parallel. Note as well that often we will use the term **orthogonal** in place of perpendicular.

Now, if two vectors are orthogonal then we know that the angle between them is 90 degrees. From (2) this tells us that if two vectors are orthogonal then,

$$\vec{a} \bullet \vec{b} = 0$$

Likewise, if two vectors are parallel then the angle between them is either 0 degrees (pointing in the same direction) or 180 degrees (pointing in the opposite direction). Once again using (2) this would mean that one of the following would have to be true.

$$\vec{a} \bullet \vec{b} = \|\vec{a}\| \, \|\vec{b}\| \; (\theta = 0°) \qquad \text{OR} \qquad \vec{a} \bullet \vec{b} = -\|\vec{a}\| \, \|\vec{b}\| \; (\theta = 180°)$$

Example 3 Determine if the following vectors are parallel, orthogonal, or neither.

(a) $\vec{a} = \langle 6, -2, -1 \rangle$, $\vec{b} = \langle 2, 5, 2 \rangle$

(b) $\vec{u} = 2\vec{i} - \vec{j}$, $\vec{v} = -\frac{1}{2}\vec{i} + \frac{1}{4}\vec{j}$

Solution

(a) First get the dot product to see if they are orthogonal.

$$\vec{a} \bullet \vec{b} = 12 - 10 - 2 = 0$$

The two vectors are orthogonal.

(b) Again, let's get the dot product first.

$$\vec{u} \bullet \vec{v} = -1 - \frac{1}{4} = -\frac{5}{4}$$

So, they aren't orthogonal. Let's get the magnitudes and see if they are parallel.

$$\|\vec{u}\| = \sqrt{5} \qquad \|\vec{v}\| = \sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4}$$

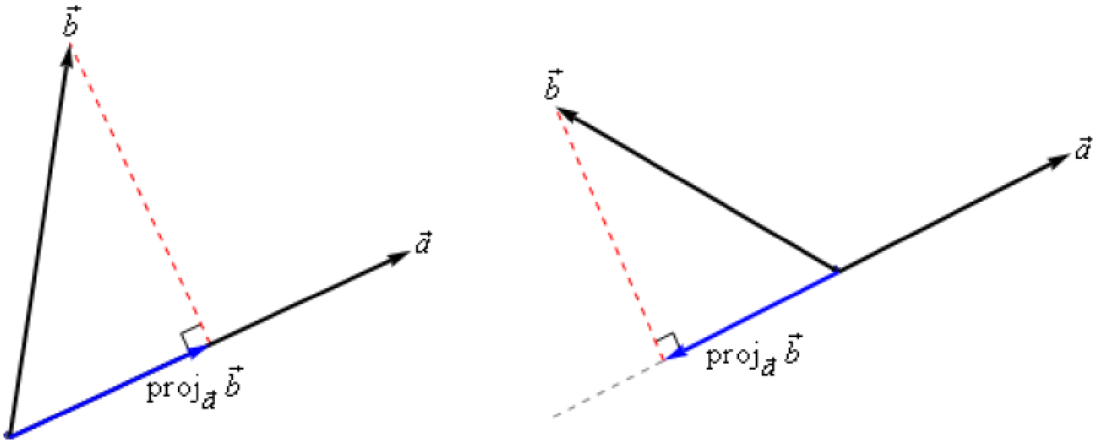
Now, notice that,

$$\vec{u} \bullet \vec{v} = -\frac{5}{4} = -\sqrt{5}\left(\frac{\sqrt{5}}{4}\right) = -\|\vec{u}\| \, \|\vec{v}\|$$

So, the two vectors are parallel.

C. <u>Projections.</u>

The best way to understand projections is to see a couple of sketches. So, given two vectors $\vec{a}$ and $\vec{b}$ we want to determine the projection of $\vec{b}$ onto $\vec{a}$. The projection is denoted by $\mathrm{proj}_{\vec{a}}\,\vec{b}$. Here are a couple of sketches illustrating the projection.



So, to get the projection of $\vec{b}$ onto $\vec{a}$ we drop straight down from the end of $\vec{b}$ until we hit (and form a right angle) with the line that is parallel to $\vec{a}$. The projection is then the vector that is parallel to $\vec{a}$, starts at the same point both of the original vectors started at and ends where the dashed line hits the line parallel to $\vec{a}$.

There is an nice formula for finding the projection of $\vec{b}$ onto $\vec{a}$. Here it is,

$$\mathrm{proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a} \bullet \vec{b}}{\left\|\vec{a}\right\|^2}\,\vec{a}$$

Note that we also need to be very careful with notation here. The projection of $\vec{a}$ onto $\vec{b}$ is given by

$$\mathrm{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a} \bullet \vec{b}}{\left\|\vec{b}\right\|^2}\,\vec{b}$$

We can see that this will be a totally different vector. This vector is parallel to $\vec{b}$, while $\mathrm{proj}_{\vec{a}}\,\vec{b}$ is parallel to $\vec{a}$. So, be careful with notation and make sure you are finding the correct projection.

Here's an example.

Example 4 Determine the projection of $\vec{b} = \langle 2,1,-1 \rangle$ onto $\vec{a} = \langle 1,0,-2 \rangle$.

Solution

We need the dot product and the magnitude of $\vec{a}$.

$$\vec{a} \bullet \vec{b} = 4 \qquad \|\vec{a}\|^2 = 5$$

The projection is then,

$$\begin{aligned} \text{proj}_{\vec{a}}\, \vec{b} &= \frac{\vec{a} \bullet \vec{b}}{\|\vec{a}\|^2} \vec{a} \\ &= \frac{4}{5} \langle 1,0,-2 \rangle \\ &= \left\langle \frac{4}{5}, 0, -\frac{8}{5} \right\rangle \end{aligned}$$

Example 5 Determine the projection of $\vec{a} = \langle 1,0,-2 \rangle$ onto $\vec{b} = \langle 2,1,-1 \rangle$.

Solution

We need the dot product and the magnitude of $\vec{b}$.

$$\vec{a} \bullet \vec{b} = 4 \qquad \|\vec{b}\|^2 = 6$$

The projection is then,

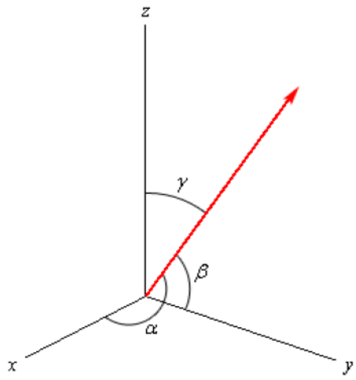
$$\begin{aligned} \text{proj}_{\vec{b}}\, \vec{a} &= \frac{\vec{a} \bullet \vec{b}}{\|\vec{b}\|^2} \vec{b} \\ &= \frac{4}{6} \langle 2,1,-1 \rangle \\ &= \left\langle \frac{4}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle \end{aligned}$$

D. <u>Direction cosine.</u>

This application of the dot product requires that we be in three dimensional space unlike all the other applications we've looked at to this point.

Let's start with a vector, $\vec{a}$, in three dimensional space. This vector will form angles with the *x*-axis (α), the *y*-axis (β), and the *z*-axis (γ). These angles are called **direction angles** and the cosines of these angles are called **direction cosines**.

Here is a sketch of a vector and the direction angles.



The formulas for the direction cosines are,

$$\cos\alpha = \frac{\vec{a}\bullet\vec{i}}{\|\vec{a}\|} = \frac{a_1}{\|\vec{a}\|} \qquad \cos\beta = \frac{\vec{a}\bullet\vec{j}}{\|\vec{a}\|} = \frac{a_2}{\|\vec{a}\|} \qquad \cos\gamma = \frac{\vec{a}\bullet\vec{k}}{\|\vec{a}\|} = \frac{a_3}{\|\vec{a}\|}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the standard basis vectors.

Let's verify the first dot product above. We'll leave the rest to you to verify.

$$\vec{a}\bullet\vec{i} = \langle a_1, a_2, a_3\rangle \bullet \langle 1, 0, 0\rangle = a_1$$

Here are a couple of nice facts about the direction cosines.

1. The vector $\vec{u} = \langle \cos\alpha, \cos\beta, \cos\gamma\rangle$ is a unit vector.
2. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
3. $\vec{a} = \|\vec{a}\|\langle \cos\alpha, \cos\beta, \cos\gamma\rangle$

Let's do a quick example involving direction cosines.

Example 6 Determine the direction cosines and direction angles for $\vec{a} = \langle 2, 1, -4\rangle$.

Solution

We will need the magnitude of the vector.

$$\|\vec{a}\| = \sqrt{4+1+16} = \sqrt{21}$$

The direction cosines and angles are then,

$$\cos\alpha = \frac{2}{\sqrt{21}} \qquad \alpha = 1.119 \text{ radians} = 64.123 \text{ degrees}$$

$$\cos\beta = \frac{1}{\sqrt{21}} \qquad \beta = 1.351 \text{ radians} = 77.396 \text{ degrees}$$

$$\cos\gamma = \frac{-4}{\sqrt{21}} \qquad \gamma = 2.632 \text{ radians} = 150.794 \text{ degrees}$$

5- ***Cross Product.***